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Master Thesis

Generalizations of the Choquet Integral Applied to Classification and Decision Making Problems

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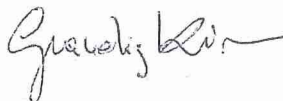
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*“If people do not believe that mathematics is simple, it is only
because they do not realize how complicated life is.”*

— JOHN VON NEUMANN

ABSTRACT

WIECZYNSKI, Jonata Cristian. **Generalizations of the Choquet Integral Applied to Classification and Decision Making Problems.** 2021. 99 p. Thesis (Master) – Programa de Pós-Graduação em Computação. Universidade Federal do Rio Grande - FURG, Rio Grande.

The discrete Choquet integral is an averaging aggregation function which is commonly used in a variety of applications. Recently, a series of generalizations of the Choquet integral have been introduced, like the CT-integrals, CC-integrals, CF-integrals and d-Choquet integral. They were applied in diverse problems and have shown good results, compared with the standard Choquet integral.

Two of those applications are classification, by using Fuzzy Rule-Based Classification Systems (FRBCS), and decision making, with the Group Modular Choquet Random Technique for Order by Preference to Ideal Solution (GMC-RTOPSIS). The FRBCS have been an extensively application of the generalizations of the Choquet integral, as they are quite explainable and their results are good. On the other hand, the GMC-RTOPSIS is a multi-criteria decision making method, which is able to process multiple types of data as input, as well as dealing with interaction among the pre-established criteria via the standard Choquet integral.

The first objective of this Master's thesis was to use the already developed theory of CT- and CC-integrals in decision making. We have applied them in the GMC-RTOPSIS in place of the standard Choquet integral. The second was to develop the concept of dCF-integral, a generalization of the Choquet integral based on functions F , from the CF-integral, and dissimilarity functions, from the d-Choquet integral. We also tested the applicability of the dCF-integral in classification and decision making problems.

Lastly, the CT- and CC-integrals showed a better separation when using functions other than the product. In both of the methods the Łukasiewicz t-norm/copula performed better in distinguishing the top of the rank. When applying the dCF-integrals in classification problems, we obtained better results for some combinations of functions F and dissimilarities than using the CF-integrals. This is also seen in the decision making method GMC-RTOPSIS, which presented a higher separation of the alternatives when using the dCF-integrals in comparison to using CT-, CC- and the standard Choquet integral.

Keywords: Choquet Integral Generalizations, Decision Making, Classification.

RESUMO

WIECZYNSKI, Jonata Cristian. **Generalizações da Integral de Choquet Aplicadas em Problemas de Classificação e Tomada de Decisão**. 2021. 99 f. Dissertação (Mestrado) – Programa de Pós-Graduação em Computação. Universidade Federal do Rio Grande - FURG, Rio Grande.

A integral de Choquet é uma função de agregação discreta, a qual é comumente utilizada em várias aplicações. Recentemente, uma série de generalizações da integral de Choquet foram apresentadas, algumas delas sendo: as CT-integrais, as CC-integrais, as CF-integrais e as d-Choquet integrais. Sendo, a maioria, aplicada em diversos problemas e, mostraram ótimos resultados quando comparadas a integral de Choquet tradicional.

Duas dessas aplicações são: a classificação, utilizando Sistemas de Classificação baseados em Regras Fuzzy (FRBCS, em inglês), e a tomada de decisão, com o *Group Modular Choquet Random Technique for Order by Preference to Ideal Solution* (GMC-RTOPSIS). Os FRBCS veem sido amplamente utilizados em generalizações da integral de Choquet, por serem explicativas e seus resultados serem bons. Por outro lado, o método GMC-RTOPSIS é um método de tomada de decisão multi critério, sendo capaz de processar múltiplos tipos de dados na entrada, bem como as interações entre os critérios pré-estabelecidos através da integral de Choquet.

O primeiro objetivo dessa dissertação de mestrado é em usar a teoria já proposta das CT- e CC-integrais na tomada de decisão. Onde aplicamos elas no método GMC-RTOPSIS no lugar da integral de Choquet tradicional. O segundo objetivo é o de desenvolver o conceito da dCF-integral, uma generalização da integral de Choquet que é baseada em funções F , usadas na CF-integral, e em funções de dissimilaridade, utilizadas na d-Choquet integral. Ainda, executa-se um teste aplicando as dCF-integrais em problemas de classificação e tomada de decisão.

Por fim, as aplicações da CT- e CC-integrais na tomada de decisão resultaram em uma melhor separação do ranking quando utilizando outras funções que não são o produto. Em ambos os casos a t-norma/copula Łukasiewicz teve uma performance superior na distinção das alternativas do topo do ranking. No caso da aplicação das dCF-integrais em classificação, obteve-se uma melhora nos resultados para algumas combinações de funções F e dissimilaridades ao comparar com o resultado do das CF-integrais. Isto também ocorre na tomada de decisão, utilizando o GMC-RTOPSIS, onde este apresentou uma melhor separação entre as alternativas ao usar as dCF-integrais em comparação a usar as CT-, CC-integral e a integral de Choquet tradicional.

Palavras-chave: Generalizações da Integral de Choquet, Tomada de Decisão, Classificação.

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LIST OF ABBREVIATIONS AND ACRONYMS

AF	Aggregation Function
DM	Decision Maker
FRM	Fuzzy Reasoning Method
FRBCS	Fuzzy Rule-Based Classification Systems
FST	Fuzzy Set Theory
GMC-RTOPSIS	Group Modular Choquet Random TOPSIS
MADM	Multi-Attribute Decision Making
MCDM	Multi-Criteria Decision Making
MODM	Multi-Objective Decision Making
NIS	Negative Ideal Solution
PAF	Pre-Aggregation Function
PIS	Positive Ideal Solution
PSO	Particle Swarm Optimization
TOPSIS	Technique for Order Preference by Similarity to Ideal Solution

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1 Introduction

Fuzzy Set Theory has been receiving a lot of attention, in regard to decision systems. First introduced by Zadeh [40], the theory has grown and its applicability in computational problems has spread widely, by providing ways of describing problems where data is imprecise or uncertain.

One of the Fuzzy tools is the Choquet integral [10], which is an integral that aggregate information [20] via non-additive measure [37] called fuzzy measures [10, 34]. The Choquet integral is an averaging aggregation operator [20], i.e., it is from a family of functions called mean functions, which contains functions that their result are always between the minimum and maximum of their respective inputs.

Recently, generalizations of the Choquet integral have been developed where the product operator is replaced by a series of more generic operators or functions. One of the proposed generalization is the C_T -integral, introduced by Lucca et al. [27]. This integral generalizes the Choquet integral by using triangular norms (t-norms, for short) [25] in place of the product. Applications of this integral were done in classification [27] and image processing [12], showing promising results.

Another generalization of the Choquet integral is the CC -integral, also developed by Lucca et al. [28]. The CC -integral is based on the expanded Choquet integral (when we use the distribution law of the product operation), and replaces the two products in the Choquet integral for copula functions [31].

The above generalizations are quite restrictive in the sense of which functions can be used as a replacement of the product, inside the integral. Therefore, an even more generic approach was used to produce the C_F -integrals [29], a family of integrals where any bivariate function F can be used.

More recently, Bustince et al. introduced a new type of integral called d-Choquet integral (also d-integral) [6]. The approach of this integral is different from the ones above, as it uses restricted dissimilarities [7] instead of the difference of the standard Choquet integral. There are no published studies showing its applications to this date.

With respect to decision making, it is strong linked to day life, not only of individuals, but also businesses and industries in general. From something as simple as choosing a

shoe to a more complex decision as selecting an appropriated supplier, decisions must be taken to reach a good equilibrium on life and business.

That reason encouraged researches to develop decision making methods [35] to mathematically analyze the alternatives, based on pre-established criteria. One class of methods are the Multi-Criteria Decision Making (MCDM), that deals with evaluating criteria in decision making. This study is based on problems that can be described in a discrete universe, commonly called Multi-Attributes Decision Making (MADM), considering that, problems that deal with continuous spaces, called Multi-Objective Decision Making (MODM), are out of the scope of this work. Additionally, MCDM and MADM will be used interchangeable as meaning problems in a discrete space.

One of the most popular MCDM methods is the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), initially developed by Huang and Yoon [21], and continually being improved as today [35]. This method uses the idea of ideal solutions, that is, the best and worst possible solution of the problem - which are called Positive and Negative Ideal Solutions (PIS and NIS), respectively - to measure their distances to each of the alternatives, and, then, rank accordingly to the highest value.

Multiple adaptations of the TOPSIS were proposed, like the Fuzzy TOPSIS [23], group TOPSIS [33] and also interval TOPSIS [18]. Lately, a new extension of the TOPSIS was developed, called Group Modular Choquet Random TOPSIS (GMC-RTOPSIS) [26], with ability to deal with groups of decision makers, multiple and heterogeneous data types, such as crisp values, fuzzy sets, probabilistic distribution, etc.; and interaction among the criteria by means of the Choquet integral.

On the other hand, Fuzzy Rule-Based Classification Systems (FRBCS) [22] have been used in various studies lately [27, 28, 29]. By using the Fuzzy Reasoning Method (FRM) [11] to learn how fuzzy rules' information will classify new samples. Most of the generalizations of the Choquet integral, such as C_T and CC , have been applied in this technique, showing excellent results.

For a introduction of the theoretical background to the theories used in this thesis see Appendix B.

1.1 Objectives and contributions

There were two main objectives in this dissertation. The first, was to use the already developed theory of C_T - and CC -integrals in decision making with the GMC-RTOPSIS method. And, the second, was subdivided in two areas: theoretical and applied. For the theoretical part we developed the theory of dC_F -integrals, that are a composition from the methodologies from Lucca et al. and Bustince et al.. In the applied part, we apply it in classification, by using FRBCS, and in decision making problems, by using GMC-RTOPSIS.

The specific objectives were:

1. Apply the C_T -integral in the GMC-RTOPSIS and compare the results from different C_T -integrals using an evaluation metric based on the separation between the alternatives ranked first and second;
2. Apply the CC -integral in the GMC-RTOPSIS and analyze the results based on the same metric used in item 1.
3. To generalize the Choquet integral using the Lucca et al. and Bustince et al. methodologies by composing the bivariate and restricted dissimilarity functions in the integral to create a new integral called dC_F -integral;
4. Apply the dC_F -integral in a classification problem and analyze its results;
5. Apply the dC_F -integral in the GMC-RTOPSIS, and analyze the results based on the metric from item 1.

In summary, the main scientific contributions of this master's thesis are:

- Develop the theory of dC_F -integrals with an application in classification;
- Apply the C_T -, C_F - and dC_F -integrals in the GMC-RTOPSIS method.

1.2 Organization of the text

The rest of this document is organized as follows: Chapter 2 presents the article entitled “Generalizing the GMC-RTOPSIS Method using CT-integral Pre-aggregation Functions”. In Chapter 3 we show the article “CC-Separation Measure Applied in Business Group Decision Making”. Then, in Chapter 4 we present the article “dCF-integrals: improving the performance and usability of CF-integrals in fuzzy rule based classification systems by a generalization based on dissimilarities”. Lastly, in Chapter 5 we apply the dCF-integrals in the GMC-RTOPSIS method and analyze the results. Finally, our conclusion is in Chapter 6.

2 Article I - Generalizing the GMC-RTOPSIS Method using CT-integral Pre-aggregation Functions

The following article [38] was published and presented at the IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2020), which composes the IEEE World Congress in Computational Intelligence (WCCI 2020)¹.

The article proposes an application of the C_T -integrals [27] in decision making, by using it instead of the Choquet integral [10] in the GMC-RTOPSIS method [26]. We defined, in the paper, the T-separation measure as the squared root of the C_T -integral to be used in the GMC-RTOPSIS algorithm.

We tested the substitution by using five different t-norms T in the C_T -integral and applying it in the “Case Study 2” from the Lourenzutti et al. paper. In the presented problem a company is analyzing suppliers, and it asks three managers to give their evaluations, in their own criteria, of each supplier. The problem is able to show the method’s capacity in dealing with multiple data types as inputs and, also, the interaction between the criteria.

Since we are working with multiple ranks, one for each t-norm used, we also proposed how to decide which result should we consider: by choosing the result with the biggest difference between the first and the second ranked alternatives. This approach was selected because when the difference is small the rankings may invert its positions. As an example: in this paper we obtained a ranking for the algebraic product t-norm, which is the standard Choquet integral, as $A_4 \geq A_3 \geq A_1 \geq A_2$, while in [26] the rank was $A_3 \geq A_4 \geq A_1 \geq A_2$ (A_i represents alternatives of the article’s described problem).

The results have shown that, by using the Łukasiewicz t-norm, the separation between the first and second alternatives rank values were the biggest, and the algebraic product, in the other hand, was the smallest.

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¹<https://wcci2020.org>

Generalizing the GMC-RTOPSIS Method using CT-integral Pre-aggregation Functions

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Abstract—In Multi-Criteria Decision Making, one of the most used algorithm designed to deal with decision making is the Technical Order by Preference to Ideal Solution (TOPSIS), which is based on finding a solution that is close to the best possible solution and distant from the worst possible solution. The Group Modular Choquet Random TOPSIS (GMC-RTOPSIS) is a generalization of the TOPSIS method capable of dealing with multiple and heterogeneous data types and interaction among criteria by means of the discrete Choquet integral. On the other hand, CT-integrals are a generalization of the Choquet integral using t-norms, which are more flexible than the standard Choquet integral. CT-integrals are pre-aggregation functions, which means that we do not require them to be monotonic in the whole domain, just in some specific directions, that is, they are directionally monotonic. Due to the excellent performance of CT-integrals in classification and multimodal fuzzy fusion decision problems, the objective of this paper is to generalize the GMC-RTOPSIS by using CT-integrals and to analyze the results provided by the use of five different t-norms in an example of a decision making problem.

Index Terms—Choquet integral, CT-integrals, pre-aggregation function, GMC-RTOPSIS, Multi-Criteria Decision Making

I. INTRODUCTION

Decision making is strongly linked with daily life, not only concerning with individuals, but also with businesses and industries, in general. From choosing something as simple as a shoe to a more complex decision of selecting an appropriated

supplier, decisions must be taken to reach a good equilibrium between what will be gained and lost in conflicting criteria.

One algorithm designed to deal with decision making is the Technical Order by Preference to Ideal Solution (TOPSIS) [1], [2]. It falls into a class of problems called Multi-Criteria Decision Making (MCDM) [3], [4] and is based on finding a solution that is closer to the best possible solution (Positive Ideal Solution (PIS)) and distant from the worst possible solution (Negative Ideal Solution (NIS)).

Lourenzutti et al. [5] introduced the Group Modular Choquet Random TOPSIS (GMC-RTOPSIS), which is a generalization of the TOPSIS method capable of dealing with multiple and heterogeneous data types (in particular, they use intuitionistic fuzzy data [6]) and interaction among criteria by means of the discrete Choquet integral [7], proposing a method based on Particle Swarm Optimization (PSO) [8] to learn the fuzzy measure.

Observe that the discrete Choquet integral, as a special aggregation function [9], is a fuzzy integral that can be used to integrate a function with respect to a (non-additive) fuzzy measure [10], and, thus, it is able to consider the influence of the interaction among the elements to be integrated (that is, to be aggregated) [11], [12]. In fact, the Choquet integral is an averaging aggregation function [13], meaning that its outputs carry information laying in a range from the minimum to the maximum of the information provided by the inputs.

The Choquet integral has been applied in different problems, such as image processing [14], [15], risk evaluation [16], models for robust sustainable development [17], classification [18], ensembles of classifiers [19], and mainly in decision

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making and multi-criteria decision making [5], [20], [21]. See the work by Dimuro et al. [12] for a discussion about the applications of the Choquet integral.

In 2016, Lucca et al. introduced a generalization of the Choquet integral using t-norms $T : [0, 1]^2 \rightarrow [0, 1]$ [22], called C_T -integrals [23], which are more flexible than the standard Choquet integral. First, C_T -integrals define a family of fuzzy integrals (which includes the standard Choquet integral) and, then, one always can choose one of them that best fit to the problem in question. Secondly, C_T -integrals are pre-aggregation functions [23], which means that they are not required to be monotonic in the whole domain, just in some specific directions, that is, they are directionally monotonic [24].

C_T -integrals have presented excellent results when used to perform the aggregation task in the fuzzy reasoning method of Fuzzy Rule-based Classification Systems [23] and also in multimodal fuzzy fusion decision [25]. For further details on the use of directional monotonicity in applications see [12].

Then, the objective of this present work is to generalize the GMC-RTOPSIS by using C_T -integrals and to analyze the results obtained by the use of five different t-norms (namely, the algebraic product – which provides the original method –, the Minimum, the Łukasiewicz t-norm, the Nilpotent product and the Hamacher product) in an example of a decision making problem.

This paper is organized as follows. In Section II it is presented the basic framework to understand the algorithm. In Section III a overview of the GMC-RTOPSIS is presented. Section IV explains the generalization of GMC-RTOPSIS by C_T -integrals. Section V shows the example problem used for simulations followed by an analysis of the obtained results. Finally, Section VI is the Conclusion.

II. PRELIMINARY CONCEPTS

In this section, we recall the preliminary concepts necessary to develop the paper.

A. Intuitionistic Fuzzy Numbers

A fuzzy set [26] a is a set defined on universe X by a membership function $\mu_a : X \rightarrow [0, 1]$, denoted by

$$a = \{ \langle x, \mu_a(x) \rangle \mid x \in X \}.$$

A fuzzy set a is a trapezoidal fuzzy number (TFN), denoted by $a = (a_1, a_2, a_3, a_4)$, where $a_1 \leq a_2 \leq a_3 \leq a_4$, if the membership function μ_a is defined on $\mathbb{R}$ as:

$$\mu_a(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & \text{if } a_1 \leq x < a_2 \\ 1, & \text{if } a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}, & \text{if } a_3 < x \leq a_4 \\ 0, & \text{otherwise.} \end{cases}$$

The distance between two TFNs $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$ is defined as:

$$d(a, b) = \sqrt{\frac{1}{4} \sum_{i=1}^4 (a_i - b_i)^2}.$$

The defuzzified value of a TFN $a = (a_1, a_2, a_3, a_4)$ is given by:

$$m(a) = \frac{a_1 + a_2 + a_3 + a_4}{4}.$$

Definition 2.1: An intuitionistic fuzzy set (IFS) [6] A is defined on a universe X by a membership function $\mu_A : X \rightarrow [0, 1]$ and a non-membership function $\nu_A : X \rightarrow [0, 1]$ such that $\mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$, that is:

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}.$$

Let $\tilde{\mu}_A$ and $\tilde{\nu}_A$ be the maximum membership degree and the minimum non-membership degree, respectively, of an intuitionistic fuzzy set (IFS) A .

Definition 2.2: An IFS A is an intuitionistic trapezoidal fuzzy number (ITFN), denoted by

$$A = \langle (a_1, a_2, a_3, a_4), \tilde{\mu}_A, \tilde{\nu}_A \rangle,$$

where $a_1 \leq a_2 \leq a_3 \leq a_4$, if μ_A and ν_A are given, for all $x \in \mathbb{R}$, by:

$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} \tilde{\mu}_A, & \text{if } a_1 \leq x < a_2 \\ \tilde{\mu}_A, & \text{if } a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3} \tilde{\mu}_A, & \text{if } a_3 < x \leq a_4 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\nu_A(x) = \begin{cases} \frac{1-\tilde{\nu}_A}{a_1-a_2} (x-a_1) + 1, & \text{if } a_1 \leq x < a_2 \\ \tilde{\nu}_A, & \text{if } a_2 \leq x \leq a_3 \\ \frac{1-\tilde{\nu}_A}{a_4-a_3} (x-a_4) + 1, & \text{if } a_3 < x \leq a_4 \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2.3: The distance between two IFS $A = \langle (a_1, a_2, a_3, a_4), \tilde{\mu}_A, \tilde{\nu}_A \rangle$ and $B = \langle (b_1, b_2, b_3, b_4), \tilde{\mu}_B, \tilde{\nu}_B \rangle$ is:

$$d(A, B) = \frac{1}{2} [d_{\tilde{\mu}}(A, B) + d_{\tilde{\nu}}(A, B)]$$

where

$$d_{\kappa}(A, B) = \left\{ \frac{1}{4} \left[(a_1 - b_1)^2 + (1 + (\kappa_A - \kappa_B))^2 (1 + (a_2 - b_2)^2 + (a_3 - b_3)^2 - 1 + (a_4 - b_4)^2) \right] \right\}^{1/2}$$

for $\kappa = \tilde{\mu}, \tilde{\nu}$.

B. Aggregation and Pre-aggregation Functions

Aggregation functions are an important class of functions that combines a n -ary input into a single output following two simple proprieties:

Definition 2.4 (Aggregation function [9]): An aggregation function is a function that maps $n > 1$ arguments onto the unit interval, i.e., $f : [0, 1]^n \rightarrow [0, 1]$ such that:

- (A1) $f(0) = 0$ and $f(1) = 1$; (Boundary conditions)
- (A2) If $\mathbf{x} \leq \mathbf{y}$ then $f(\mathbf{x}) \leq f(\mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in [0, 1]^n$. (Monotonicity)

TABLE I: T-norms used in this paper

t-norms	Definition
Algebraic product	$T_p(x, y) = xy$
Minimum	$T_M(x, y) = \min\{x, y\}$
Łukasiewicz	$T_L(x, y) = \max\{0, x + y - 1\}$
Nilpotent product	$T_{NM}(x, y) = \begin{cases} \min\{x, y\}, & \text{if } x + y > 1 \\ 0, & \text{otherwise} \end{cases}$
Hamacher product	$T_{HP}(x, y) = \begin{cases} 0, & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy}, & \text{otherwise} \end{cases}$

An important kind of aggregation function are the t-norms.

Definition 2.5 (Triangular norm [22]): A triangular norm (or t-norm) is a bi-variate aggregation function $T : [0, 1]^2 \rightarrow [0, 1]$, such that, for all $x, y, z \in [0, 1]$, T satisfies the following properties:

- (T1) $T(x, y) = T(y, x)$; (Commutativity)
- (T2) $T(x, T(y, z)) = T(T(x, y), z)$; (Associativity)
- (T3) $T(x, y) \leq T(x, z)$ if $y \leq z$; (Monotonicity)
- (T4) $T(x, 1) = x$. (Boundary condition)

The t-norms adopted in this paper are shown in Table I.

In order to generalize earlier work on relaxed monotonicity constraints, to encompass non-monotonic functions within the aggregation framework, Bustince et. al. [24] defined directional monotonicity as follows:

Definition 2.6 (Directional Monotonicity): Let $\mathbf{r}$ be a real n -dimensional vector, such that $\mathbf{r} \neq \mathbf{0}$. A function $f : [0, 1]^n \rightarrow [0, 1]$ is said to be $\mathbf{r}$ -increasing if, for all $\mathbf{x} \in [0, 1]^n$ and for all $c > 0$ such that $\mathbf{x} + c\mathbf{r} \in [0, 1]^n$, it holds that

$$f(\mathbf{x} + c\mathbf{r}) \geq f(\mathbf{x}).$$

Considering the directional monotonicity, Lucca et al. [23] introduced the concept of pre-aggregation functions:

Definition 2.7 (Pre-aggregation function [23]): A function $PA : [0, 1]^n \rightarrow [0, 1]$ is said to be an n -ary pre-aggregation function if the following condition holds:

- (PA1) There exists $\mathbf{r} \in [0, 1]^n$, $\mathbf{r} \neq \mathbf{0}$, such that PA is $\mathbf{r}$ -increasing; (Directional Monotonicity)
- (PA2) $PA(\mathbf{0}) = 0$ and $PA(\mathbf{1}) = 1$. (Boundary Conditions)

C. Choquet Integral

The Choquet integral is defined considering the concept of fuzzy measure and can be used to model the interaction among its arguments.

Definition 2.8 (Fuzzy measure [27]): Let N be a finite set and 2^N be its power set. Then a function $\varphi : 2^N \rightarrow [0, 1]$ is a fuzzy measure if, for all $X, Y \subseteq N$, the following conditions holds:

- 1) $\varphi(\emptyset) = 0$ and $\varphi(X) = 1$;
- 2) if $X \subseteq Y$, then $\varphi(X) \leq \varphi(Y)$.

Definition 2.9 (Choquet integral [7]): Let φ be a fuzzy measure. Then the Choquet integral of $\mathbf{x} \in [0, 1]^n$ with respect to φ is $\mathcal{C}_\varphi[0, 1]^n \rightarrow [0, 1]$, defined by:

$$\mathcal{C}_\varphi(\mathbf{x}) = \sum_{i=1}^n [x_{(i)} - x_{(i-1)}] \varphi(A_{(i)})$$

where (i) represents a permutation on 2^N such that $x_{(i-1)} \leq x_{(i)}$ for all $i = 1, \dots, n$, with $x_{(0)} = 0$ and $A_{(i)} = \{(1), \dots, (i)\}$.

The product used in the Choquet integral is, in fact, the algebraic product t-norm (see Table I). Lucca et al. [23] generalized the concept of Choquet integral by substituting this operator for others from the t-norm family, which created a new class of fuzzy integral, named $\mathcal{C}_T$ -integrals by Dimuro et al. [12]:

Definition 2.10 ($\mathcal{C}_T$ -integral [12], [23]): Let φ be a fuzzy measure and $T : [0, 1]^2 \rightarrow [0, 1]$ a t-norm. Then a $\mathcal{C}_T$ -integral is defined as $\mathcal{C}_\varphi^T : [0, 1]^n \rightarrow [0, 1]$, given, for all $\mathbf{x} \in [0, 1]^n$, by

$$\mathcal{C}_\varphi^T(\mathbf{x}) = \sum_{i=1}^n T(x_{(i)} - x_{(i-1)}, \varphi(A_{(i)}))$$

where (i) represents a permutation on 2^N such that $x_{(i-1)} \leq x_{(i)}$ for all $i = 1, \dots, n$, with $x_{(0)} = 0$ and $A_{(i)} = \{(1), \dots, (i)\}$.

In Lucca et al. [23] it was proved that $\mathcal{C}_T$ -integrals are pre-aggregation functions that are not, in general, aggregation functions. In fact, $\mathcal{C}_T$ -integrals are $(k, \dots, k)$ -increasing, for any $k \geq 1$, that is, $\mathcal{C}_T$ -integrals are weakly monotonic [28], which is a particular case of directional monotonicity.

III. DECISION MAKING AND GMC-RTOPSIS

The GMC-RTOPSIS [5] is a method introduced to allow multiple data types to be processed in the decision matrix. It uses the Choquet integral [7] to aggregate the criteria according to their interactions based on the fuzzy measure provided and/or simulated.

The decision matrix for the GMC-RTOPSIS method is:

$$DM^d = \begin{matrix} & C_1 & C_2 & \dots & C_{n_d} \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} & \begin{pmatrix} s_{11}^d(\mathbf{Y}^d) & s_{12}^d(\mathbf{Y}^d) & \dots & s_{1n_d}^d(\mathbf{Y}^d) \\ s_{21}^d(\mathbf{Y}^d) & s_{22}^d(\mathbf{Y}^d) & \dots & s_{2n_d}^d(\mathbf{Y}^d) \\ \vdots & \vdots & \ddots & \vdots \\ s_{m1}^d(\mathbf{Y}^d) & s_{m2}^d(\mathbf{Y}^d) & \dots & s_{mn_d}^d(\mathbf{Y}^d) \end{pmatrix} \end{matrix}$$

where:

- 1) Each decision maker $d \in \{1, \dots, k\}$, with $k \in \mathbb{N}$, is represented by one matrix;
- 2) The set $\mathbf{A} = \{A_1, \dots, A_m\}$ is the alternative set, with $m \in \mathbb{N}$, which is the same for all decision makers;
- 3) The set $\mathbf{C}_d = \{C_1, \dots, C_{n_d}\}$, with $n_d \in \mathbb{N}$, is the criteria set for the decision maker d . Additionally, the set $\mathbf{C} = \{C_1, \dots, C_k\} = \{C_1, \dots, C_n\}$, where $n = \sum_{d=1}^k n_d$, represents the criteria for all decision makers;

- 4) Each value $s_{ij}^d(\mathbf{Y}^d)$, with $1 \leq i \leq m$ and $1 \leq j \leq n_d$, is called the rating of the criterion j for alternative i ;
- 5) The vector $\mathbf{Y} = (\mathbf{Y}_{rand}, \mathbf{Y}_{det})$ is a vector that represents random events and deterministic ones.

After all decision makers have provided their evaluations for a problem in question, then the decision algorithm can be used. The process follows the same structure of the original TOPSIS [1] with one difference: each criterion uses a different distance measure, since each one may contain a different data type. Therefore, each criterion distance is calculated separately and aggregated in the separation measure step using the Choquet integral.

The Positive Ideal Solution (PIS) and the Negative Ideal Solution (NIS) are, respectively, the one that is closer to the best possible solution and the one that is distant from the best possible solution.

The algorithm can be summarized as follows [5]:

- 1) Normalize all matrices;
- 2) Select the PIS, denoted by $s_j^+(\mathbf{Y})$, and the NIS, denoted by $s_j^-(\mathbf{Y})$, considering, for each $j \in \{1, \dots, n\}$, respectively:

$$s_j^+(\mathbf{Y}) = \begin{cases} \max_{1 \leq i \leq m} s_{ij}, & \text{if it is a benefit criterion,} \\ \min_{1 \leq i \leq m} s_{ij}, & \text{if it is a cost/loss criterion,} \end{cases}$$

$$s_j^-(\mathbf{Y}) = \begin{cases} \min_{1 \leq i \leq m} s_{ij}, & \text{if it is a benefit criterion,} \\ \max_{1 \leq i \leq m} s_{ij}, & \text{if it is a cost/loss criterion;} \end{cases}$$

- 3) Calculate the distance measure for each criterion C_j , with $j \in \{1, \dots, n\}$, to the PIS and NIS solutions, that is,

$$d_{ij}^+ = d(s_j^+(\mathbf{Y}), s_{ij}(\mathbf{Y})),$$

$$d_{ij}^- = d(s_j^-(\mathbf{Y}), s_{ij}(\mathbf{Y})),$$

where $i \in \{1, \dots, m\}$.

- 4) Calculate the separation measure, for each $i \in \{1, \dots, m\}$, using the Choquet integral as follows:

$$S_i^+(\mathbf{Y}) = \sqrt{\sum_{j=1}^n \left(d_{i(j)}^+\right)^2 \left(\varphi_{\mathbf{Y}}(C_{(j)}^+) - \varphi_{\mathbf{Y}}(C_{(j+1)}^+)\right)}$$

$$S_i^-(\mathbf{Y}) = \sqrt{\sum_{j=1}^n \left(d_{i(j)}^-\right)^2 \left(\varphi_{\mathbf{Y}}(C_{(j)}^-) - \varphi_{\mathbf{Y}}(C_{(j+1)}^-)\right)}$$

where $d_{i(1)}^+ \leq \dots \leq d_{i(n)}^+$, $d_{i(1)}^- \leq \dots \leq d_{i(n)}^-$, for each $j \in \{1, \dots, n-1\}$, $C_{(j)}^+$ is the criterion correspondent to $d_{i(j)}^+$, $C_{(j)}^-$ is the criterion correspondent to $d_{i(j)}^-$, $C_{(j)}^+ = \{C_{(j)}^+, C_{(j+1)}^+, \dots, C_{(n)}^+\}$, $C_{(j)}^- = \{C_{(j)}^-, C_{(j+1)}^-, \dots, C_{(n)}^-\}$, $C_{(n+1)}^+ = \emptyset$ and $C_{(n+1)}^- = \emptyset$. Note that the separation measures are the square root of the Choquet integral of squared distances and thus is a d-Choquet integral – a Choquet integrals based on dissimilarities – introduced by Bustince et al. [29]. Observe also that the fuzzy measure may be dependent

on $\mathbf{Y}_{det}$, that is, in each state one may consider a different fuzzy measure.

- 5) For each $i \in \{1, \dots, m\}$, calculate the relative closeness coefficient to the ideal solution with:

$$CC_i(\mathbf{Y}) = \frac{S_i^-(\mathbf{Y})}{S_i^-(\mathbf{Y}) + S_i^+(\mathbf{Y})};$$

- 6) Since the CC_i values may be a bootstrapped probability distribution we need to use a point $\widehat{cc}_i$ that better describes this distribution. Thus, the rank of the alternatives is done by minimizing a pre-defined risk function:

$$\widehat{cc}_i = \arg \min_c R(c)$$

$$= \arg \min_c \int_{\mathbb{R}} L(c, CC_i(\mathbf{Y})) dF(CC_i(\mathbf{Y})); \quad (1)$$

- 7) Obtain the final estimation $\widehat{cc}_i = f_{x \in \mathcal{X}}(\widehat{cc}_i)$, where f is an aggregation function that assembles the $\widehat{cc}_i$ values from all states $x \in \mathcal{X}$.

IV. GENERALIZING THE GMC-RTOPSIS BY C_T -INTEGRALS

The GMC-RTOPSIS uses the Choquet integral in the separation measure step. This study generalizes this method for working with the generalization of the Choquet integral by t-norms, called C_T -integrals, which are pre-aggregation functions but not aggregation functions. In particular, here we use the t-norms presented in Table I.

Definition 4.1 (T-separation measure): Let $T : [0, 1]^2 \rightarrow [0, 1]$ be a t-norm. A T -separation measure is defined, for each $i \in \{1, \dots, m\}$, by the functions:

$$S_i^+(\mathbf{Y}) = \sqrt{\sum_{j=1}^n T\left(\left(d_{i(j)}^+\right)^2, \varphi_{\mathbf{Y}}(C_{(j)}^+) - \varphi_{\mathbf{Y}}(C_{(j+1)}^+)\right)}$$

$$S_i^-(\mathbf{Y}) = \sqrt{\sum_{j=1}^n T\left(\left(d_{i(j)}^-\right)^2, \varphi_{\mathbf{Y}}(C_{(j)}^-) - \varphi_{\mathbf{Y}}(C_{(j+1)}^-)\right)}$$

where $d_{i(j)}^+$, $d_{i(j)}^-$, $C_{(j)}^+$, $C_{(j)}^-$ and $\varphi_{\mathbf{Y}}$ are defined as in item 4 above. Note that the separation measure is the squared root of the C_T -integral, which is a pre-aggregation function as shown in [23]. Additionally, since the square root function is strictly increasing on $[0, 1]$, the separation measure is also a pre-aggregation function.

V. APPLICATION EXAMPLE OF GMC-RTOPSIS GENERALIZED BY C_T -INTEGRALS

For the sake of comparison, we use the case study 2 presented in [5].

A. Methodology

The simulations were performed with 10,000 samples from the DM matrix and applied a Particle Swarm Optimization (PSO) [8] algorithm to determine the best fuzzy measure, with 20 iterations and 30 particles. Since the GMC-RTOPSIS used the PSO algorithm and have shown good results, the method

TABLE II: Linguistic variable and their respective trapezoidal fuzzy numbers

Linguistic variables	Trapezoidal fuzzy numbers
Worst (W)	(0, 0, 0.2, 0.3)
Poor (P)	(0.2, 0.3, 0.4, 0.5)
Intermediate (I)	(0.4, 0.5, 0.6, 0.7)
Good (G)	(0.6, 0.7, 0.8, 1)
Excellent (E)	(0.8, 0.9, 1, 1)

presented here also use the same algorithm to learn the fuzzy measure.

Additionally, for ranking the alternatives the squared loss was used:

$$L(cc, CC_i) = (cc - CC_i)^2$$

in (1). This gives the mean as a point estimator for the ranking process.

Note that the algorithm complexity is the same as the original GMC-RTOPSIS method for each t-norm.

B. Example Description

Suppose a company is evaluating four suppliers for a determined provision, i.e., A_1, A_2, A_3 and A_4 . To analyze those suppliers the company let three managers give their ratings based on their criteria for each of the A_i alternatives.

The budget manager considered the price per batch (in thousands), warranty (in days) and payment conditions (in days), represented as criterion $C_1^{(1)}, C_2^{(1)}$ and $C_3^{(1)}$, respectively. Additionally, suppose that the demand of the product is higher in December. To model this into the problem, let τ be a binary variable such that $\tau = 0$ when the month is between January and November and $\tau = 1$ when it is December. The weighting vector given by this manager was

$$\mathbf{w}^{(1)} = (0.5, 0.25, 0.25).$$

The production manager examined the price, delivery time (in hours), production capacity, product quality and the time to respond to a support request (in hours), representing criteria $C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)}$ and $C_5^{(2)}$ respectively. Furthermore, this manager wants to account for the reliability of the production process and what a failure in the process can cause in the production capacity of the supplier. So, let P_i be a random variable such that $P_i = 0$ occurs when there are no failures in the production process of the supplier A_i , and $P_i = 1$ when there are failures. Also, when December arrives the production is much higher, so the probability of failure is higher also. Lastly, the higher demand in December and the issues in the production process affect the delivery time, so let it be modeled by the following ITFN:

$$\begin{aligned} s_{13}^2 &= ((0.8^{1+P_1}, 0.9^{1+P_1}, 1.0^{1+P_1}, 1.0^{1+P_1}), 1.0, 0.0) \\ s_{23}^2 &= ((0.8^{1+4P_2}, 0.9^{1+4P_2}, 1.0^{1+4P_2}, 1.0^{1+4P_2}), 0.7, 0.1) \\ s_{33}^2 &= ((0.6^{1+2P_3}, 0.7^{1+2P_3}, 0.8^{1+2P_3}, 1.0^{1+2P_3}), 0.8, 0.0) \\ s_{43}^2 &= ((0.5^{1+3P_4}, 0.6^{1+3P_4}, 0.8^{1+3P_4}, 0.9^{1+3P_4}), 0.8, 0.1) \end{aligned}$$

and consider

$$f_i(x, y) = x(1 + y(P_i + \tau)^2).$$

The weighting vector for the criteria used by this manager is

$$\mathbf{w}^{(2)} = (0.2, 0.2, 0.2, 0.2, 0.2).$$

The commercial manager considered the product lifespan (in years), social and environmental responsibility, the quantity of quality certifications and price, representing criteria $C_1^{(3)}, C_2^{(3)}, C_3^{(3)}$ and $C_4^{(3)}$ respectively. The weighting vector provided by this manager is

$$\mathbf{w}^{(3)} = (0.25, 0.12, 0.23, 0.4).$$

Considering all the DM s, we have the following underlying factors: a random component $\mathbf{Y}_{rand} = (P_1, P_2, P_3, P_4)$ and a deterministic component $Y_{det} = \tau$ that has two states: S_1 when $\tau = 0$ and S_2 when $\tau = 1$. The underlying factors can be represented by $\mathbf{Y} = (\mathbf{Y}_{rand}, \mathbf{Y}_{det})$. Since the state S_2 was considered more important, it was given a higher weight in the decision: $w(S_1) = 0.4$ and $w(S_2) = 0.6$. Additionally, the company assigned the weighting vector $\mathbf{w} = (0.3, 0.4, 0.3)$ for the decision makers. The DM matrices for each manager are presented in Table III, where the linguistic variables (W, P, I, G and E) are defined as Table II.

The P_i distribution were determined by historical data of each supplier and are:

- For $\tau = 0$:

$$\begin{aligned} p(P_1 = 0|S_1) &= 0.98, \\ p(P_2 = 0|S_1) &= 0.96, \\ p(P_3 = 0|S_1) &= 0.97, \\ p(P_4 = 0|S_1) &= 0.95. \end{aligned}$$

- For $\tau = 1$:

$$\begin{aligned} p(P_1 = 0|S_2) &= 0.96, \\ p(P_2 = 0|S_2) &= 0.92, \\ p(P_3 = 0|S_2) &= 0.96, \\ p(P_4 = 0|S_2) &= 0.90. \end{aligned}$$

Furthermore, the company wanted to include some interaction between the criteria, so a variation of 30% was allowed for each fuzzy measure. This measure is calculated computationally by means of the PSO algorithm [5].

C. Results

The summary of the results using each one of the t-norms is listed in Table V for State 1 and Table VI for State 2 with the best result being in **boldface**. It can be seen that when using the algebraic product in the Choquet integral we have the following ranking

$$A_4^{Pr}(0.5508) > A_3^{Pr}(0.5475) > A_1^{Pr}(0.4469) > A_2^{Pr}(0.4323).$$

Also, when the minimum t-norm is applied the alternatives maintain the same rank as when using the algebraic product:

$$A_4^{\min}(0.5373) > A_3^{\min}(0.5087) > A_1^{\min}(0.4544) > A_2^{\min}(0.4302).$$

TABLE III: Decision matrices for the managers

(a) Budget manager					
Alternatives	$C_1^{(1)}$	$C_2^{(1)}$	$C_3^{(1)}$		
			$\tau = 0$	$\tau = 1$	
A_1	$260.00(1 + 0.15\tau)$	90	G	G	
A_2	$250.00(1 + 0.25\tau)$	90	P	W	
A_3	$350.00(1 + 0.20\tau)$	180	G	I	
A_4	$550.00(1 + 0.10\tau)$	365	I	W	

(b) Production manager					
Alternatives	$C_1^{(2)}$	$C_2^{(2)}$	$C_3^{(2)}$	$C_4^{(2)}$	$C_5^{(2)}$
A_1	260.00	$U(f_1(48, 0.10), f_1(96, 0.10))$	s_{13}^2	I	[24, 48]
A_2	250.00	$U(f_2(72, 0.20), f_2(120, 0.20))$	s_{23}^2	P	[24, 48]
A_3	350.00	$U(f_3(36, 0.15), f_3(72, 0.15))$	s_{33}^2	G	[12, 36]
A_4	550.00	$U(f_4(48, 0.25), f_4(96, 0.25))$	s_{34}^2	E	[0, 24]

(c) Commercial manager				
Alternatives	$C_1^{(3)}$	$C_2^{(3)}$	$C_3^{(3)}$	$C_4^{(3)}$
A_1	Exp(3.5)	W	1	260.00
A_2	Exp(3.0)	W	0	250.00
A_3	Exp(4.5)	P	3	350.00
A_4	Exp(5.0)	I	5	550.00

However, when the Łukasiewicz t-norm is substituted in the Choquet integral the top of the ranking switch positions:

$$A_3^{\mathbf{L}}(0.6028) > A_4^{\mathbf{L}}(0.5158) > A_1^{\mathbf{L}}(0.4963) > A_2^{\mathbf{L}}(0.4628).$$

The same ranking appears when using the nilpotent product t-norm, with

$$A_3^N(0.5843) > A_4^N(0.5252) > A_1^N(0.4798) > A_2^N(0.4591)$$

and the Hamacher product with

$$A_3^H(0.5411) > A_4^H(0.5386) > A_1^H(0.4567) > A_2^H(0.4345).$$

When analyzing a decision making problem, in order to determine which ranking result to adopt among the different results provided by the use of several t-norms, we propose to calculate the distance between the alternative ranked in the first place and the one ranked in second place. This may indicate a better discrimination between the alternatives, that is, a guarantee that there is no doubt about which is the best alternative to be chosen.

In the example presented before, the t-norm that best discriminates the first two ranked options is the Łukasiewicz t-norm with a distance of 0.0870. Additionally, the t-norm that least discriminates the two first ranked alternatives is the Hamacher product with a distance of 0.0025, that is, the two first ranked alternatives are too close. The distances calculated for each t-norm is shown in Table IV, where the best result is marked in **boldface** and the worst in underline.

TABLE IV: Distance between the first ranked and the second ranked alternative

t-norm	Distance
Algebraic product	0.0033
Minimum	0.0286
Łukasiewicz	0.0870
Nilpotent product	0.0591
Hamacher product	<u>0.0025</u>

VI. CONCLUSION

This paper introduced a generalization of GMC-RTOPSIS by using C_T -integrals, with an example application in decision making process.

The GMC-RTOPSIS method is a generalization of the classical TOPSIS, with capacity of dealing with multiple and heterogeneous data types and, also, interacting criteria using the discrete Choquet integral. For these reasons, the applications of the method are substantial.

Since the GMC-RTOPSIS uses the discrete Choquet integral, it can be broadened to a wide generalization by using C_T -integrals, which has been already applied and shown its efficiency on other problems. The process is done by substituting the original algebraic product on the Choquet integral by different t-norms.

Observe that this generalization provides more flexibility and confidence in the process, since, by varying the used t-norm, one obtains a solution set that confirm the ranking of alternatives. Moreover, one can choose the solution that presents the highest separation between the ranking values. This is the case, for example, of the Łukasiewicz t-norm

TABLE V: Mean and standard deviation values for state 1 (S_1 , $\tau = 0$) using different t-norms

CC	Algebraic product		Minimum		Łukasiewicz product		Nilpotent product		Hamacher product	
	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
A_1	0.4482	0.0050	0.4311	0.0118	0.4699	0.0312	0.4843	0.0240	0.4493	0.0128
A_2	0.4514	0.0044	0.4363	0.0106	0.4591	0.0140	0.4777	0.0114	0.4487	0.0093
A_3	0.5584	0.0212	0.5039	0.0331	0.5741	0.0224	0.5820	0.0341	0.5429	0.0300
A_4	0.5464	0.0040	0.5584	0.0114	0.5346	0.0165	0.5197	0.0099	0.5462	0.0099

TABLE VI: Mean and standard deviation values for state 2 (S_2 , $\tau = 1$) using different t-norms

CC	Algebraic product		Minimum		Łukasiewicz product		Nilpotent product		Hamacher product	
	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
A_1	0.4460	0.0123	0.4699	0.0153	0.5139	0.0089	0.4768	0.0302	0.4617	0.0154
A_2	0.4195	0.0103	0.4261	0.0105	0.4653	0.0129	0.4467	0.0182	0.4251	0.0135
A_3	0.5402	0.0217	0.5119	0.0365	0.6219	0.0258	0.5858	0.0341	0.5399	0.0338
A_4	0.5537	0.0096	0.5233	0.0135	0.5032	0.0130	0.5289	0.0179	0.5336	0.0133

and the Nilpotent product, which presents the largest and second largest difference between the first and the second rank. Clearly, in this example, one should adopt the ranking result provided by the Łukasiewicz t-norm.

It is important to note that, currently, does not exist a metric capable of guarantee that the presented method is the best in all possible problems. Although, with the use of multiple Choquet integrals one can use the t-norm which better discriminates the alternatives.

Further work will consider other kinds of generalizations of the Choquet integral, like CC-integrals [30], C_F -integrals [31], $C_{F_1 F_2}$ -integrals [32] and $gC_{F_1 F_2}$ -integrals [33]. In particular, a C_F -Integral generalizes the work in this paper, where instead of using t-norms we use a class of function F satisfying certain conditions. For more information on the generalizations of the Choquet integral and their applications, see [12], [34]–[36].

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3 Article II - CC-Separation Measure Applied in Business Group Decision Making

In this article we used the CC-integrals [28] in the separation measure of the GMC-RTOPSIS method [26] instead of the standard Choquet integral [10]. The CC-integrals is a family of Choquet-like integrals that uses copulas functions in place of the product operation in the expanded Choquet integral.

We defined the CC-separation measure, which is the squared root of the CC-integral and applied it in the same problem presented in the previews paper [38] and in [26]. Twelve different copula functions were used; for the overlap alpha function, two different parameters were used $\alpha = 0.1$ and $\alpha = 0.6$, since the first one is from the literature [28] and the second one was the one that better performed in this problem.

A note on the comparisons with the previews article about the C_T : for the functions that can be used in both C_T and CC-integrals, we can not compare them directly since most of them do not hold the distributive property, i.e., $\min\{x_{(i)} - x_{(i-1)}, m(A_{(i)})\} \neq \min\{x_{(i)}, m(A_{(i)})\} - \min\{x_{(i-1)}, m(A_{(i)})\}$. The exception is the algebraic product, which gives the standard Choquet integral. Although, we can compare them with the resulting rankings via the difference between the first alternatives values.

The results obtained, for the presented problem, show that the Łukasiewicz copula (also a t-norm) has the greatest separation between the alternatives, followed by the overlap alpha with parameter 0.6, the copula F and the Nilpotent copula (also t-norm). Lastly, the algebraic product scored one of the lowest separations, showing that it may exchange the positions of the best placed alternatives.

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CC-Separation Measure Applied in Business Group Decision Making

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Abstract: In business, one of the most important management functions is decision making. The Group Modular Choquet Random TOPSIS (GMC-RTOPSIS) is a Multi-Criteria Decision Making (MCDM) method that can work with multiple heterogeneous data types. This method uses the Choquet integral to deal with the interaction between different criteria. The Choquet integral has been generalized and applied in various fields of study, such as imaging processing, brain-computer interface, and classification problems. By generalizing the so-called extended Choquet integral by copulas, the concept of CC-integrals has been introduced, presenting satisfactory results when used to aggregate the information in Fuzzy Rule-Based Classification Systems. Taking this into consideration, in this paper, we applied 11 different CC-integrals in the GMC-RTOPSIS. The results demonstrated that this approach has the advantage of allowing more flexibility and certainty in the choosing process by giving a higher separation between the first and second-ranked alternatives.

1 INTRODUCTION

Business managers rely on the right decisions to keep their business competitive. Many times a decision has to be made by multiple analysts and considering various criteria. This is a time consuming and expensive task. Although, most of the time, it can be solved by an algorithm or mathematical model, like route, supplier chain, and location problems (Deveci et al., 2017; Alazzawi and Žak, 2020; Shyur and Shih, 2006), releasing the pressure of the decision from the managers, and allow them to work on other processes of the company/industry.

The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) (Huang and Yoon, 1981) is one of the multi-criteria decision making (MCDM) methods that ranks the best possible solu-

tion among a set of alternatives. This approach is based on pre-defined criteria, using the alternative's distance to the best and worst possible solutions for the problems, Positive and Negative Ideal Solutions (PIS and NIS), respectively.

In 2017, the Group Modular Choquet Random TOPSIS (GMC-RTOPSIS) (Lourenzutti et al., 2017) was introduced. The method generalized the original TOPSIS allowing it to deal with multiple and heterogeneous data types. The approach models the interaction among the criteria by using the discrete Choquet integral (Choquet, 1954). The Choquet integral allows a function to be integrated by using non-additive fuzzy measures (Choquet, 1954; Candeloro et al., 2019), which means that it can consider the interaction among the elements that are being integrated (Murofushi and Sugeno, 1989; Dimuro et al., 2020). The GMC-RTOPSIS learns the fuzzy measure associated with the criteria with a Particle Swarm Optimization (PSO) algorithm (Wang et al., 2011).

The C_T -integrals (Lucca et al., 2016) is a generalization of the Choquet integral that replaces the product operation by triangular norm (t-norm) functions (Klement et al., 2011). The C_T -integrals are a family

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of integrals that are pre-aggregation functions (Lucca et al., 2016). Additionally, C_T -integrals are averaging functions, i.e., the result is always between the minimum and maximum of the input.

The T-separation measure (Wieczynski et al., 2020) was introduced and applied in the GMC-RTOPSIS instead of the Choquet integral. In this study, the authors considered five different T-separation measures to tackle Case Study 2 from (Lourenzutti et al., 2017). The problem consists of choosing a new supplier for a company by asking various decision-makers to give their opinions with different criteria. The problem is posed with a variety of data types, such as probability distributions, fuzzy numbers, and interval numbers. The paper also proposed to use the t-norm that better discriminates the first ranked alternative to the second one by calculating the difference of the rankings. The approach presented good results when using the Łukasiewicz t-norm (T_L), giving a better separation between the ranked alternatives than the standard Choquet integral.

After introducing the C_T -integrals, Lucca et al. have proposed the CC-integrals (Lucca et al., 2017a). CC-integrals are a generalization of the Choquet integral in its expanded form, satisfying some properties, such as averaging, idempotency, and aggregation (Grabisch et al., 2009). The authors applied the CC-integral in classification problems, showing that the function based on the minimum is the one that produced the highest performance of the classifier. The CC-integrals have been studied in the literature by Dimuro et al., where the properties of CMin integrals (Lucca et al., 2017b; Dimuro et al., 2018; Mesiar and Stupňanová, 2019) were analyzed.

In this paper, we define the CC-separation measure and use it in the GMC-RTOPSIS instead of the Choquet integral. We also apply our approach in an application as an example, the same used in (Lourenzutti et al., 2017; Wieczynski et al., 2020). Finally, we analyze its results and compare them to other works.

The paper is organized as follows: Section 2 introduces the basic concepts about the fuzzy set theory and TOPSIS decision making. In Section 3 we define the CC-separation measure. In Section 4 we show our methodology, the problem used to test the CC-separation measure, and the results achieved by using it. Lastly, the conclusion is in Section 5.

2 PRELIMINARY CONCEPTS

In this section, we recall the preliminary concepts necessary to develop the paper.

2.1 Fuzzy Set Theory

A Fuzzy Set (Zadeh, 1965) is defined on a universe X by a membership function $\mu_a : X \rightarrow [0, 1]$, denoted by

$$a = \{ \langle x, \mu_a(x) \rangle \mid x \in X \}.$$

We call a trapezoidal fuzzy number (TFN) the fuzzy set denoted by $a = (a_1, a_2, a_3, a_4)$, where $a_1 \leq a_2 \leq a_3 \leq a_4$, if the membership function μ_a is defined on $\mathbb{R}$ as:

$$\mu_a(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & \text{if } a_1 \leq x < a_2 \\ 1, & \text{if } a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}, & \text{if } a_3 < x \leq a_4 \\ 0, & \text{otherwise.} \end{cases}$$

A measure of the distance between two TFNs $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$ is defined as:

$$d(a, b) = \sqrt{\frac{1}{4} \sum_{i=1}^4 (a_i - b_i)^2}.$$

The defuzzified value of a TFN $a = (a_1, a_2, a_3, a_4)$ is given by:

$$m(a) = \frac{a_1 + a_2 + a_3 + a_4}{4}.$$

An intuitionistic fuzzy set (IFS) A is defined on a universe X by a membership function $\mu_A : X \rightarrow [0, 1]$ and a non-membership function $\nu_A : X \rightarrow [0, 1]$ such that $\mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$, that is:

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}.$$

Let $\tilde{\mu}_A$ and $\tilde{\nu}_A$ be the maximum membership degree and the minimum non-membership degree, respectively, of an IFS A .

An IFS A is an intuitionistic trapezoidal fuzzy number (ITFN), denoted by

$$A = \langle (a_1, a_2, a_3, a_4), \tilde{\mu}_A, \tilde{\nu}_A \rangle$$

where $a_1 \leq a_2 \leq a_3 \leq a_4$, if μ_A and ν_A are given, for all $x \in \mathbb{R}$, by

$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} \tilde{\mu}_A, & \text{if } a_1 \leq x < a_2 \\ \tilde{\mu}_A, & \text{if } a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3} \tilde{\mu}_A, & \text{if } a_3 < x \leq a_4 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_A(x) = \begin{cases} \frac{1-\tilde{\nu}_A}{a_1-a_2} (x-a_1) + 1, & \text{if } a_1 \leq x < a_2 \\ \tilde{\nu}_A, & \text{if } a_2 \leq x \leq a_3 \\ \frac{1-\tilde{\nu}_A}{a_4-a_3} (x-a_4) + 1, & \text{if } a_3 < x \leq a_4 \\ 1, & \text{otherwise.} \end{cases}$$

The distance between two ITFNs $A = \langle (a_1, a_2, a_3, a_4), \tilde{\mu}_A, \tilde{\nu}_A \rangle$ and $B = \langle (b_1, b_2, b_3, b_4), \tilde{\mu}_B, \tilde{\nu}_B \rangle$ is:

$$d(A, B) = \frac{1}{2} [d_{\tilde{\mu}}(A, B) + d_{\tilde{\nu}}(A, B)]$$

where

$$d_{\kappa}(A, B) = \left\{ \frac{1}{4} \left[(a_1 - b_1)^2 + (1 + (\kappa_A - \kappa_B)^2) \right. \right. \\ \left. \left. (1 + (a_2 - b_2)^2 + (a_3 - b_3)^2) \right. \right. \\ \left. \left. - 1 + (a_4 - b_4)^2 \right] \right\}^{1/2}$$

for $\kappa_A = \tilde{\mu}_A$ and $\kappa_B = \tilde{\mu}_B$ when $\kappa = \mu$; and for $\kappa_A = \tilde{\nu}_A$ and $\kappa_B = \tilde{\nu}_B$ when $\kappa = \nu$.

Aggregation functions (**AF**) (Grabisch et al., 2009) are used to unify inputs into a single value representing them all and are defined as a function that maps $n > 1$ arguments onto the unit interval, that is, a function $f : [0, 1]^n \rightarrow [0, 1]$ such that the boundaries, $f(\mathbf{0}) = 0$ and $f(\mathbf{1}) = 1$, with $\mathbf{0}, \mathbf{1} \in [0, 1]^n$, and the monotonicity properties, $\mathbf{x} \leq \mathbf{y} \implies f(\mathbf{x}) \leq f(\mathbf{y})$, $\forall \mathbf{x}, \mathbf{y} \in [0, 1]^n$, hold.

A triangular norm (t-norm) is an aggregation function $T : [0, 1]^2 \rightarrow [0, 1]$ that satisfies, for any $x, y, z \in [0, 1]$: the commutative and associative properties and the boundary condition.

An overlap function (Bustince et al., 2010) $O : [0, 1]^2 \rightarrow [0, 1]$ is a function that satisfies the following conditions:

- O is commutative;
- $O(x, y) = 0 \iff xy = 0$;
- $O(x, y) = 1 \iff xy = 1$;
- O is increasing;
- O is continuous.

A bivariate function $Co : [0, 1]^2 \rightarrow [0, 1]$ is called a copula (Nelsen, 2007) if, for all $x, x', y, y' \in [0, 1]$ with $x \leq x'$ and $y \leq y'$, the following conditions hold:

- $Co(x, y) + Co(x', y') \geq Co(x, y') + Co(x', y)$;
- $Co(x, 0) = Co(0, x) = 0$;
- $Co(x, 1) = Co(1, x) = x$.

The Choquet integral is defined based on a fuzzy measure (Sugeno, 1974), that is, a function m from the power set of N to the unit interval, $m : 2^N \rightarrow [0, 1]$, that for all $X, Y \subset N$ holds the conditions:

- (1) $m(\emptyset) = 0$ and $m(N) = 1$;
- (2) if $X \subset Y$, then $m(X) \leq m(Y)$.

From this, Choquet defined the integral as: Let m be a fuzzy measure. The Choquet integral (Choquet, 1954) of $\mathbf{x} \in [0, 1]^n$ with respect to m is defined as:

$$\mathfrak{C}_m : [0, 1]^n \rightarrow [0, 1] \\ \mathbf{x} \rightarrow \sum_{i=1}^n (x_{(i)} - x_{(i-1)}) m(A_{(i)})$$

where (i) is a permutation on 2^N such that $x_{(i-1)} \leq x_{(i)}$ for all $i = 1, \dots, n$, with $x_{(0)} = 0$ and $A_{(i)} = \{(1), \dots, (i)\}$.

Notice that one can use the distributive law to expand the Choquet integral into:

$$\mathfrak{C}_m = \sum_{i=1}^n (x_{(i)} m(A_{(i)}) - x_{(i-1)} m(A_{(i)})) \quad (1)$$

Recently, the Choquet integral was generalized by copula functions. By substituting the product operator by copulas in the expanded form of the Choquet integral (Eq. 1), CC-Integrals (Lucca et al., 2017a) were introduced.

Let m be a fuzzy measure and Co be a bivariate copula. The Choquet-like integral based on copula with respect to m is defined as a function $\mathfrak{C}_m^{Co} : [0, 1]^n \rightarrow [0, 1]$, for all $\mathbf{x} \in [0, 1]^n$, by

$$\mathfrak{C}_m^{Co} = \sum_{i=1}^n Co(x_{(i)}, m(A_{(i)})) - Co(x_{(i-1)}, m(A_{(i)})) \quad (2)$$

where (i) , $x_{(i)}$ and $A_{(i)}$ is defined as the Choquet integral.

It is important to note that the Choquet integral, the C_T -integrals, and the CC -integrals are averaging functions, i.e., the results from them are always bounded by the minimum and maximum of their input.

2.2 Decision Making

The GMC-RTOPSIS (Lourenzutti et al., 2017) is a decision making algorithm that improved the classic TOPSIS (Huang and Yoon, 1981) by allowing groups of decision-makers, modularity in the input, multiple input types and, by using the Choquet integral, the ability to measure the interaction among different criteria.

Figure 1 shows an overview of the decision making process with the Choquet integral. Here three different decision-makers give their ratings for three products based on three criteria. These ratings are then processed and inserted in the Choquet integral, where the interaction between the criteria is calculated. After, the results are ranked according to their highest classiness coefficient value.

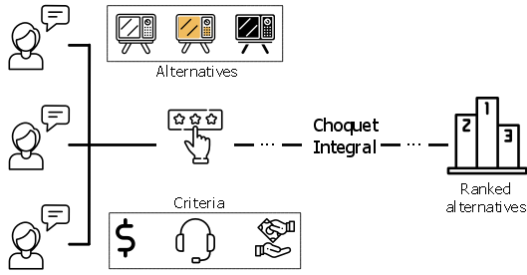


Figure 1: Image description of the decision making process using the Choquet integral. Source: the authors.

To describe the GMC-RTOPSIS method let d represent the d -th decision maker in a collection of $k \in \mathbb{N}$ ones. Let $\mathbf{A} = \{A_1, \dots, A_m\}$ be the set of alternatives for the problem and $\mathbf{C}_d = \{C_1, \dots, C_{n_d}\}$ represent the criteria set for decision maker d . With $\mathbf{C} = \{\mathbf{C}_1, \dots, \mathbf{C}_k\} = \{C_1, \dots, C_n\}$, where $n = \sum_{d=1}^k n_d$, representing the criteria set of all the decision makers. From these notations we can represent each of the d -th decision maker by the matrix below (Eq. (3)), called decision matrix DM:

$$DM^d = \begin{matrix} & C_1 & C_2 & \dots & C_{n_d} \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} & \begin{pmatrix} s_{11}^d(\mathbf{Y}^d) & s_{12}^d(\mathbf{Y}^d) & \dots & s_{1n_d}^d(\mathbf{Y}^d) \\ s_{21}^d(\mathbf{Y}^d) & s_{22}^d(\mathbf{Y}^d) & \dots & s_{2n_d}^d(\mathbf{Y}^d) \\ \vdots & \vdots & \ddots & \vdots \\ s_{m1}^d(\mathbf{Y}^d) & s_{m2}^d(\mathbf{Y}^d) & \dots & s_{mn_d}^d(\mathbf{Y}^d) \end{pmatrix} \end{matrix} \quad (3)$$

Each matrix cell $s_{ij}^d(\mathbf{Y}^d)$, with $1 \leq i \leq m$, $1 \leq j \leq n_d$, is called the rating of the criterion j for alternative i . Also, notice that the rating is a function of $\mathbf{Y} = (\mathbf{Y}_{rand}, \mathbf{Y}_{det})$, which are factors that model random and deterministic events. Random events are modeled by stochastic processes, and deterministic are events which are not random, like time, location or a parameter of a random event. A fixed value x of the deterministic vector is called a state, and the set of all states is represented by $\mathcal{X}$.

In possession of all decision matrices from all decision-makers D , the algorithm can be applied. The process is quite similar to the original TOPSIS, presented in 1981. It uses the same definition of Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS) that are, respectively, the one that is closer to the best possible solution and the one that is distant from the best possible solution, see Eq. (4). The most significant difference is that each criterion may use a different distance measure since each may have its own type. So, the distances of each criterion are calculated separately and aggregated afterward in the

separation measure step of the algorithm (see Figure 2).

In order to ease the comprehension of our approach, we present in Figure 2 the steps of the GMC-RTOPSIS, where:

Step 0. Select a state $x \in \mathcal{X}$ not yet processed;

Step 1. Normalize all matrices;

Step 2. Select the PIS, denoted by $s_j^+(\mathbf{Y})$, and the NIS, denoted by $s_j^-(\mathbf{Y})$, considering, for each $j \in \{1, \dots, n\}$, respectively:

$$s_j^+(\mathbf{Y}) = \begin{cases} \max_{1 \leq i \leq m} s_{ij}, & \text{if it is a benefit criterion,} \\ \min_{1 \leq i \leq m} s_{ij}, & \text{if it is a cost/loss criterion,} \end{cases} \quad (4)$$

$$s_j^-(\mathbf{Y}) = \begin{cases} \min_{1 \leq i \leq m} s_{ij}, & \text{if it is a benefit criterion,} \\ \max_{1 \leq i \leq m} s_{ij}, & \text{if it is a cost/loss criterion;} \end{cases}$$

Step 3. Calculate the distance measure for each criterion C_j , with $j \in \{1, \dots, n\}$, to the PIS and NIS solutions, that is,

$$d_{ij}^+ = d(s_j^+(\mathbf{Y}), s_{ij}(\mathbf{Y})),$$

$$d_{ij}^- = d(s_j^-(\mathbf{Y}), s_{ij}(\mathbf{Y})),$$

where $i \in \{1, \dots, m\}$ and d is a distance measure associated with the criteria data type;

Step 4. Calculate the separation measure, for each $i \in \{1, \dots, m\}$, using the Choquet integral as follows:

$$S_i^+(\mathbf{Y}) = \sqrt{\sum_{j=1}^n \left((d_{i(j)}^+)^2 - (d_{i(j-1)}^+)^2 \right) m_{\mathbf{Y}}(\mathbf{C}_{(j)}^+)}$$

$$S_i^-(\mathbf{Y}) = \sqrt{\sum_{j=1}^n \left((d_{i(j)}^-)^2 - (d_{i(j-1)}^-)^2 \right) m_{\mathbf{Y}}(\mathbf{C}_{(j)}^-)}$$

where $d_{i(1)}^+ \leq \dots \leq d_{i(n)}^+$, $d_{i(1)}^- \leq \dots \leq d_{i(n)}^-$, for each $j \in \{1, \dots, n-1\}$, $\mathbf{C}_{(j)}^+$ is the criterion correspondent to $d_{i(j)}^+$, $\mathbf{C}_{(j)}^-$ is the criterion correspondent to $d_{i(j)}^-$, $\mathbf{C}_{(j)}^+ = \{C_{(j)}^+, C_{(j+1)}^+, \dots, C_{(n)}^+\}$, $\mathbf{C}_{(j)}^- = \{C_{(j)}^-, C_{(j+1)}^-, \dots, C_{(n)}^-\}$, $\mathbf{C}_{(n+1)}^+ = \mathbf{C}_{(n+1)}^- = \emptyset$ and $m_{\mathbf{Y}}$ is the learned fuzzy measure by a particle swarm optimization algorithm (Wang et al., 2011).

Here, the separation measure is the square root of the Choquet integral of squared distances, and this means that it is the square root of a d-Choquet integral (Bustince et al., 2020). Also, for each state, we may have a different fuzzy measure, which means that the fuzzy measure is dependent on $\mathbf{Y}_{det}$

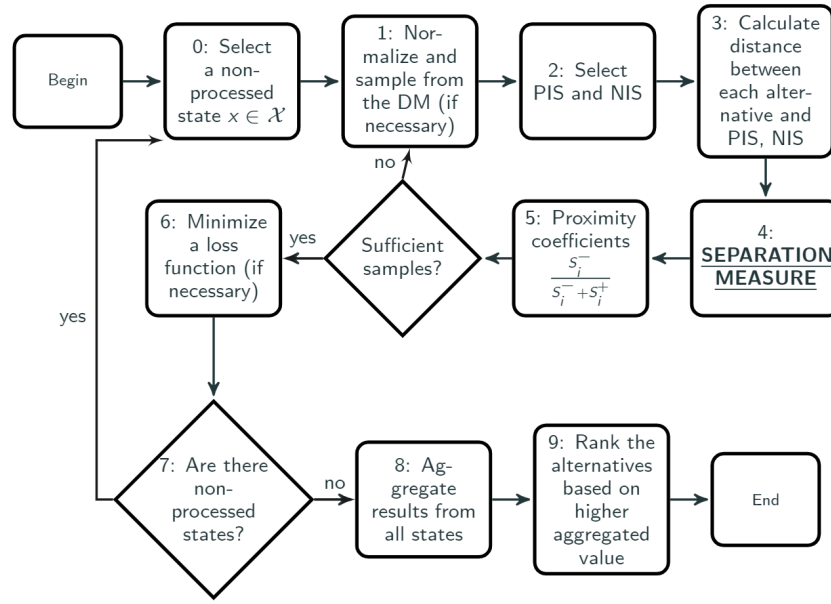


Figure 2: Diagram of the GMC-RTOPSIS process. The separation measure step is where the CC-separation measure is used. Source: The authors.

Step 5. For each $i \in \{1, \dots, m\}$, calculate the relative closeness coefficient to the ideal solution with:

$$CC_i(\mathbf{Y}) = \frac{S_i^-(\mathbf{Y})}{S_i^-(\mathbf{Y}) + S_i^+(\mathbf{Y})};$$

Step 6. By using probability distributions in the DM, it is introduced a bootstrapped probability distribution in the CC_i values, so as a point representation for this distribution we minimize a pre-defined risk function:

$$\begin{aligned} cc_i &= \arg \min_c R(c) \\ &= \arg \min_c \int_{\mathbb{R}} L(c, CC_i(\mathbf{Y})) dF(CC_i(\mathbf{Y})); \quad (5) \end{aligned}$$

Step 7. If there is at least one non-processed state x , return to Step 0;

Step 8. Aggregate the cc_i values from all the states with $\widehat{cc}_i = f_{x \in X}(cc_i(x))$, where f is an aggregation function.

Step 9. Finally, rank the alternatives from the highest to the lowest $\widehat{cc}_i$ values.

this study incorporates the CC-integrals in place of the Choquet integral.

We introduce the CC-separation measure by:

Definition 3.1 (CC-separation measure). Let Co be a bivariate copula and m a fuzzy measure. A CC-separation measure $S^* : [0, 1]^2 \rightarrow [0, 1]$ is defined, for all $i \in \{1, \dots, m\}$, by the functions:

$$\begin{aligned} S_i^+(\mathbf{Y}) &= \left[\sum_{j=1}^n Co \left(\left(d_{i(j)}^+ \right)^2, m_{\mathbf{Y}}(\mathbf{C}_{(j)}^+) \right) \right. \\ &\quad \left. - Co \left(\left(d_{i(j-1)}^+ \right)^2, m_{\mathbf{Y}}(\mathbf{C}_{(j)}^+) \right) \right]^{1/2} \\ S_i^-(\mathbf{Y}) &= \left[\sum_{j=1}^n Co \left(\left(d_{i(j)}^- \right)^2, m_{\mathbf{Y}}(\mathbf{C}_{(j)}^-) \right) \right. \\ &\quad \left. - Co \left(\left(d_{i(j-1)}^- \right)^2, m_{\mathbf{Y}}(\mathbf{C}_{(j)}^-) \right) \right]^{1/2} \end{aligned}$$

3 GENERALIZATION OF THE GMC-RTOPSIS BY USING CC-INTEGRALS

Using the Choquet integral in the separation measure, the GMC-RTOPSIS method allows for interaction among different criteria. This is the step where

where $d_{i(j)}^+$, $d_{i(j)}^-$, $\mathbf{C}_{(j)}^+$, $\mathbf{C}_{(j)}^-$ and $m_{\mathbf{Y}}$ are defined as in Step 4 of the GMC-RTOPSIS algorithm. Note that the separation measure is the squared root of the CC-integral, which is an aggregation function as shown in (Lucca et al., 2017a).

4 EXPERIMENTAL FRAMEWORK

In this section, we present the application of the CC-separations in the GMC-RTOPSIS. To do so, we start describing the methodology adopted in the study; after that, the example in which we apply our approach is described, and lastly, the obtained results are presented and discussed.

4.1 Methodology

In this study, we will apply the proposed CC-separation measure to the Case Study 2 introduced in (Lourenzutti et al., 2017) and used in (Wieczynski et al., 2020) to ease the comparison between the different CC-integrals.

To perform the simulation, we used 10,000 samples from the DM. We also applied a particle swarm optimization to learn the fuzzy measure using 30 particles and 100 interactions. The PSO is used since the original method had good outcomes with the method.

In this study we used the copula functions from Table 1. We highlight that we used 2 different values for the α parameter, which are $\alpha = 0.1$, selected from the literature (Lucca et al., 2017a), and $\alpha = 0.6$, which was the best fit for the problem presented in this paper based on the difference between the first and second ranked alternatives, when compared to other α values.

For the risk function, given in Eq. (5)), it was used the squared loss:

$$L(cc, CC_i) = (cc - CC_i)^2.$$

This result in the mean function being the point estimator for the process.

Also, we use the aggregation function bellow in the Step 8 of the algorithm:

$$WAM_i = w(S_1) \cdot cc_i(S_1) + w(S_2) \cdot cc_i(S_2).$$

For the analysis of the results from the different copula functions, we use the difference between the alternative ranked first to the second ranked with:

$$\Delta_{R1,R2} = \max(\hat{c}_1) - \max(\hat{c}_2)$$

where $\hat{c}_1 = \{cc_i \mid i \in \{1, \dots, m\}\}$ and $\hat{c}_2 = \hat{c}_1 - \{\max(\hat{c}_1)\}$.

Lastly, since we are altering only the Choquet function, the algorithm maintains its original complexity.

4.2 The considered problem

A company needs a new supplier for a provision and is evaluating four different suppliers, namely A_1 , A_2 ,

A_3 and A_4 . The company called three of its managers to analyze the suppliers and give their ratings based on their criteria.

The first manager is a budget manager. He considered the price per batch (in thousands) as $C_1^{(1)}$, warranty (in days) as $C_2^{(1)}$ and payment conditions (in days) as $C_3^{(1)}$. Also, it was considered that the demand for the product is higher in December. He modeled it by using a binary variable τ , that is $\tau = 0$ when the month is between January and November, and $\tau = 1$ when it is December. Finally, he assigned a weight for each of his criterion with a weighting vector: $\mathbf{w}^{(1)} = (0.5, 0.25, 0.25)$.

The second manager, a product manager, considered the price as $C_1^{(2)}$, delivery time (in hours) as $C_2^{(2)}$, production capacity $C_3^{(2)}$, product quality $C_4^{(2)}$ and the time to respond to a support request (in hours) as $C_5^{(2)}$. Additionally, to account for the reliability in the production process and what a failure in the process could cause to the supplier's production capacity, he let P_i be a random variable such that $P_i = 0$ occurs when there are no failures in the production process of the supplier A_i , and $P_i = 1$ when there are failures. Also, in December, the production is accelerated, so the chance of failure is higher, so he modeled a stochastic process with the help of the function:

$$f_i(x, y) = x(1 + y(P_i + \tau)^2).$$

Lastly, the production capacity was modeled by using ITFNs:

$$\begin{aligned} s_{13}^2 &= ((0.8^{1+P_1}, 0.9^{1+P_1}, 1.0^{1+P_1}, 1.0^{1+P_1}), 1.0, 0.0) \\ s_{23}^2 &= ((0.8^{1+4P_2}, 0.9^{1+4P_2}, 1.0^{1+4P_2}, 1.0^{1+4P_2}), 0.7, 0.1) \\ s_{33}^2 &= ((0.6^{1+2P_3}, 0.7^{1+2P_3}, 0.8^{1+2P_3}, 1.0^{1+2P_3}), 0.8, 0.0) \\ s_{43}^2 &= ((0.5^{1+3P_4}, 0.6^{1+3P_4}, 0.8^{1+3P_4}, 0.9^{1+3P_4}), 0.8, 0.1). \end{aligned}$$

This manager selected the same weight for all criteria, i.e $\mathbf{w}^{(2)} = (0.2, 0.2, 0.2, 0.2, 0.2)$.

The commercial manager was the third. He considered the product lifespan (in years) as $C_1^{(3)}$, social and environmental responsibility as $C_2^{(3)}$, the quantity of quality certifications as $C_3^{(3)}$ and the price as $C_4^{(3)}$. The weighting vector provided by this manager is $\mathbf{w}^{(3)} = (0.25, 0.12, 0.23, 0.4)$.

The P_i distribution was determined by historical data of each supplier and is given as follows:

- For $\tau = 0$:

$$\begin{aligned} p(P_1 = 0|S_1) &= 0.98, \\ p(P_2 = 0|S_1) &= 0.96, \\ p(P_3 = 0|S_1) &= 0.97, \\ p(P_4 = 0|S_1) &= 0.95. \end{aligned}$$

Table 1: Examples of Copulas

(I) T-norms	
Definition	Name/Description
$T_M(x, y) = \min\{x, y\}$	Minimum
$T_P(x, y) = xy$	Algebraic Product
$T_L(x, y) = \max\{0, x + y - 1\}$	Łukasiewicz
$T_{NM}(x, y) = \begin{cases} \min\{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise} \end{cases}$	Nilpotent Minimum
$T_{HP}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise} \end{cases}$	Hamacher Product
(II) Non-associative overlap functions	
Definition	Reference/Description
$O_B(x, y) = \min\{x\sqrt{y}, y\sqrt{x}\}$	Cuadras-Augé family of copulas (Nelsen, 2007)
$O_{mM}(x, y) = \min\{x, y\} \max\{x^2, y^2\}$	(Dimuro and Bedregal, 2014; Pereira Dimuro et al., 2016)
$O_\alpha(x, y) = xy(1 + \alpha(1-x)(1-y))$, where $\alpha \in [-1, 0[\cup]0, 1]$	(Alsina et al., 2006; Lucca et al., 2015)
(III) Non-associative copulas, which are neither t-norms nor overlap functions	
Definition	Reference/Description
$C_F(x, y) = xy + x^2y(1-x)(1-y)$	(Klement et al., 2011)
$C_L(x, y) = \max\{\min\{x, \frac{y}{2}\}, x + y - 1\}$	(Alsina et al., 2006)
$C_{Div}(x, y) = \frac{xy + \min\{x, y\}}{2}$	(Alsina et al., 2006)

- For $\tau = 1$:

$$\begin{aligned}
p(P_1 = 0|S_2) &= 0.96, \\
p(P_2 = 0|S_2) &= 0.92, \\
p(P_3 = 0|S_2) &= 0.96, \\
p(P_4 = 0|S_2) &= 0.90.
\end{aligned}$$

Considering all the *DMs*, we have the following underlying factors: a random component $\mathbf{Y}_{rand} = (P_1, P_2, P_3, P_4)$ and a deterministic component $Y_{det} = \tau$ that has two states: S_1 when $\tau = 0$ and S_2 when $\tau = 1$. The underlying factors can be represented by $\mathbf{Y} = (\mathbf{Y}_{rand}, \mathbf{Y}_{det})$. The managers agreed that the state S_2 was more important, since the production is higher, so they gave it a higher weight for it in the aggregation step (Step 8 of the method) by setting $w(S_1) = 0.4$ and $w(S_2) = 0.6$.

The *DMs* of all managers are presented in Table 2, where the linguistic variables (W, P, I, G and E) are defined as in Table 3.

The company, considering the opinion of manager 2 more important, assigned a weighting vector for the managers represented by $\mathbf{w} = (0.3, 0.4, 0.3)$. Fur-

thermore, they wanted to include some interaction between the criteria, so a variation of 30% was allowed for each fuzzy measure in relation to the coefficient in the additive fuzzy measure. This measure is calculated computationally by means of the PSO algorithm (Wang et al., 2011; Lourenzutti et al., 2017).

4.3 Obtained results

The aggregated ranked results are presented in Table 4 (for all the results see APPENDIX Table 5). The table shows for each copula function Co , the rank of alternatives from columns 2 to 5, with its aggregated values inside parenthesis. The last column shows the difference between alternatives ranked first and ranked second.

We can see that for the t-norms the values are proportional to the ones presented in the study that used C_T -integral instead of the Choquet integral (Wieczynski et al., 2020). As in that paper, here the T_L t-norm has the greatest difference, with $\Delta_{R1, R2} = 0.0700$. Although, only the T_{MN} t-norm performs well compared with other copulas such as O_α and C_F .

Table 2: Decision matrices for the managers
(a) Budget manager

Alternatives	$C_1^{(1)}$	$C_2^{(1)}$	$C_3^{(1)}$	
			$\tau = 0$	$\tau = 1$
A_1	$260.00(1 + 0.15\tau)$	90	G	G
A_2	$250.00(1 + 0.25\tau)$	90	P	W
A_3	$350.00(1 + 0.20\tau)$	180	G	I
A_4	$550.00(1 + 0.10\tau)$	365	I	W

(b) Production manager

Alternatives	$C_1^{(2)}$	$C_2^{(2)}$	$C_3^{(2)}$	$C_4^{(2)}$	$C_5^{(2)}$
A_1	260.00	$U(f_1(48, 0.10), f_1(96, 0.10))$	s_{13}^2	I	[24, 48]
A_2	250.00	$U(f_2(72, 0.20), f_2(120, 0.20))$	s_{23}^2	P	[24, 48]
A_3	350.00	$U(f_3(36, 0.15), f_3(72, 0.15))$	s_{33}^2	G	[12, 36]
A_4	550.00	$U(f_4(48, 0.25), f_4(96, 0.25))$	s_{34}^2	E	[0, 24]

(c) Commercial manager

Alternatives	$C_1^{(3)}$	$C_2^{(3)}$	$C_3^{(3)}$	$C_4^{(3)}$
A_1	Exp(3.5)	W	1	260.00
A_2	Exp(3.0)	W	0	250.00
A_3	Exp(4.5)	P	3	350.00
A_4	Exp(5.0)	I	5	550.00

Table 3: Linguistic variables and their respective trapezoidal fuzzy numbers

Linguistic variables	Trapezoidal fuzzy numbers
Worst (W)	(0, 0, 0.2, 0.3)
Poor (P)	(0.2, 0.3, 0.4, 0.5)
Intermediate (I)	(0.4, 0.5, 0.6, 0.7)
Good (G)	(0.6, 0.7, 0.8, 1)
Excellent (E)	(0.8, 0.9, 1, 1)

The copula O_α with $\alpha = 0.6$ achieved the second greatest difference among the tested ones, with $\Delta_{R1, R2} = 0.0502$. The next of this family tested was the one with $\alpha = 0.1$, as it is the main value used in the literature, it resulted in a quite lower difference value, with only $\Delta_{R1, R2} = 0.0294$. Among the copulas from the α family, we have C_F , with $\Delta_{R1, R2} = 0.0466$ and T_{MN} with $\Delta_{R1, R2} = 0.0454$.

Notice that when using C_{div} , C_L and T_M copulas the alternatives A_3 and A_4 change position. This is from the influence of the state 2 result, where these functions may have weighted higher criteria from alternative A_4 . Furthermore, the relative small difference $\Delta_{R1, R2}$ make the top of the rank prone to invert positions.

Our last analysis considered the CC-separations that presented the lowest ranks. Precisely, it is observable in the obtained results that the copulas T_{HP} , T_P , O_B and T_M obtained a similar performance and the lowest separation.

5 CONCLUSIONS

The GMC-RTOPSIS is a decision method that chooses the alternative that is closer to an ideal solution. It is capable of dealing with multiple data types as inputs and, also, through the Choquet integral, considers the interaction among different criteria.

In this paper, we presented the CC-separation measure. A new measure to be used in the GMC-RTOPSIS method that utilizes the CC-integrals instead of the Choquet integral. The CC-integrals is a generalization of the Choquet integral that presented good results when applied in classification problems.

By using an example from the literature, we tested the method with 11 different copula functions, with one of them using two distinct parameters. The results indicate that the Łukasiewicz t-norm is the best copula function to use in this example problem since

Table 4: Rank of the alternatives with each of the C functions, ordered by difference between the value of the alternative ranked first and second.

Co function	Ranked 1st	Ranked 2nd	Ranked 3rd	Ranked 4th	$\Delta_{R1,R2}$
T_L	$A_3(0.6462)$	$A_4(0.5762)$	$A_1(0.4616)$	$A_2(0.3782)$	0.0700
$O_{\alpha=0.6}$	$A_3(0.5897)$	$A_4(0.5395)$	$A_1(0.4716)$	$A_2(0.4282)$	0.0502
C_F	$A_3(0.5991)$	$A_4(0.5525)$	$A_1(0.4453)$	$A_2(0.4194)$	0.0466
T_{NM}	$A_3(0.5919)$	$A_4(0.5493)$	$A_1(0.4713)$	$A_2(0.3910)$	0.0425
$O_{\alpha=0.1}$	$A_3(0.5953)$	$A_4(0.5659)$	$A_1(0.4453)$	$A_2(0.3962)$	0.0294
O_{mM}	$A_3(0.5995)$	$A_4(0.5715)$	$A_1(0.4454)$	$A_2(0.3927)$	0.0280
C_{Div}	$A_4(0.5234)$	$A_3(0.5016)$	$A_1(0.4868)$	$A_2(0.4250)$	0.0218
C_L	$A_4(0.5273)$	$A_3(0.5097)$	$A_1(0.4914)$	$A_2(0.4361)$	0.0176
T_{HP}	$A_3(0.5351)$	$A_4(0.5221)$	$A_1(0.5049)$	$A_2(0.4308)$	0.0131
T_P	$A_3(0.5821)$	$A_4(0.5701)$	$A_1(0.4346)$	$A_2(0.3977)$	0.0120
O_B	$A_3(0.5511)$	$A_4(0.5395)$	$A_1(0.4713)$	$A_2(0.4133)$	0.0116
T_M	$A_4(0.5229)$	$A_3(0.5118)$	$A_1(0.4737)$	$A_2(0.4386)$	0.0110

it gives the greatest separation between the alternatives ranked first and second. Additionally, the Overlap alpha family, with $\alpha = 0.6$, the C_F and the T_{NM} also presented good separations.

By being able to verify the separation between the ranks, we can choose more confidently the alternative that better suits the problem. Therefore, by using multiple functions in the CC-separation measure, we can see how the problem behaves in different situations.

Finally, future work will consider learning the α parameter of the overlap alpha family, using distinct optimization methods.

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APPENDIX

Table 5: Mean and standard deviation of the alternatives for State 1 and State 2. The highest mean for each function and state is in **boldface** and the highest for the criterion has an asterisk*.

Alternatives Function	State 1 ($S_1, \tau = 0$)								State 2 ($S_2, \tau = 1$)							
	A_1		A_2		A_3		A_4		A_1		A_2		A_3		A_4	
	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
T_L	0.4702	0.0482	0.4360	0.0116	0.6438*	0.0367	0.5588	0.0119	0.4558	0.0506	0.3397	0.0334	0.6478*	0.0477	0.5878*	0.0250
$O_{\alpha=0.6}$	0.4852	0.0134	0.4791*	0.0232	0.5742	0.0731	0.5078	0.0108	0.4625	0.0266	0.3943	0.0165	0.6001	0.0593	0.5607	0.0214
C_F	0.4774	0.0181	0.4604	0.0173	0.6074	0.0858	0.5270	0.0135	0.4239	0.0194	0.3921	0.0208	0.5935	0.0710	0.5695	0.0174
T_{NM}	0.4643	0.0269	0.4276	0.0178	0.5939	0.0469	0.5654	0.0186	0.4759	0.0399	0.3666	0.0362	0.5905	0.0379	0.5386	0.0368
$O_{\alpha=0.1}$	0.4568	0.0127	0.4282	0.0117	0.6013	0.0791	0.5656	0.0098	0.4377	0.0275	0.3749	0.0223	0.5913	0.0611	0.5661	0.0232
O_{NM}	0.4519	0.0134	0.4218	0.0171	0.6119	0.0772	0.5668	0.0156	0.4411	0.0308	0.3733	0.0150	0.5912	0.0789	0.5746	0.0271
C_{Div}	0.5100	0.0101	0.4654	0.0187	0.5139	0.0202	0.5221	0.0191	0.4713	0.0293	0.3980	0.0088	0.4934	0.0242	0.5242	0.0307
C_L	0.5251*	0.0082	0.4670	0.0144	0.5335	0.0058	0.5242	0.0126	0.4690	0.0427	0.4155	0.0165	0.4938	0.0186	0.5293	0.0314
T_{HP}	0.4976	0.0083	0.4648	0.0141	0.5328	0.0169	0.5253	0.0125	0.5097*	0.0198	0.4081	0.0114	0.5367	0.0207	0.5199	0.0157
T_P	0.4567	0.0120	0.4270	0.0093	0.5976	0.0655	0.5674*	0.0098	0.4198	0.0396	0.3782	0.0190	0.5718	0.0605	0.5719	0.0360
O_B	0.4843	0.0148	0.4443	0.0138	0.5584	0.0422	0.5487	0.0101	0.4626	0.0360	0.3926	0.0156	0.5462	0.0388	0.5334	0.0306
T_M	0.4701	0.0478	0.4615	0.0217	0.5326	0.0018	0.5279	0.0231	0.4761	0.0270	0.4234*	0.0232	0.4980	0.0087	0.5195	0.0269

4 Article III - dC_F -integrals: An alternative of Choquet based functions based on dissimilarities

In this article we introduce a new generalization of the Choquet integral by using the methodologies from the C_F -integrals presented by Lucca et al. [29] and d-Choquet integrals, by Bustince et al. [6]. This new generalization, called dC_F -integrals, uses a generalized composition of functions [3] with a bivariate function F and the restricted dissimilarity function δ introduced by Bustince et al. [7].

Throughout the paper we prove basic properties such as aggregation [20], pre-aggregation [27], idempotency and averaging [20], exemplifying integrals with such properties. We then move to an application of the dC_F -integral in classification, more explicitly in a Fuzzy Reasoning Method (FRM) of Fuzzy Rule-Based Classification Systems (FRBCSs) [22].

The results have shown that the dC_F -integral can be considered a good alternative when compared to using the C_F -integral, which uses the difference dissimilarity. Especially when using the squared root of the difference and squared difference of the square root dissimilarities, and the F_{GL} and F_{NA} as functions F .

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dC_F -integrals: improving the performance and usability of C_F -integrals in fuzzy rule-based classification systems by a generalization based on restricted dissimilarity functions

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Abstract—The Choquet integral (CI) is an averaging aggregation function that has been used in the Fuzzy Reasoning Method (FRM) of Fuzzy Rule-Based Classification Systems (FRBCSs) and in multi-criteria decision making in order to take into account the interactions among data/criteria. Several generalizations of the CI have been proposed in the literature in order to improve the performance of FRBCSs, and also providing more flexibility in the model by relaxing both the monotonicity requirement and averaging conditions of aggregation functions. An important generalization are the CF-integrals, which are pre-aggregation functions that may present interesting non-averaging behaviour depending on the function F adopted in the construction, and, in this case, offering excellent results in classification. Recently, the concept of d-Choquet integrals was introduced as a generalization of the Choquet integral by restricted dissimilarity functions (RDFs), improving the usability of CIs, as when comparing inputs by the usual difference may not be viable. The objective of this paper is to introduce the concept of dCF-integrals, which is a generalization of CF-integrals by RDFs. The aim is to analyse whether the usage of dCF-integrals in the FRM of FRBCSs represents a good alternative towards the standard CF-integrals that just consider the difference as a dissimilarity measure. For that, we consider six different RDFs combined with five fuzzy measures, obtaining more than twenty dCF-integrals. The analysis of the results are based on statistical tests, demonstrating their efficiency. Additionally, comparing the applicability of dCF-integrals versus CF-integrals, the range of the good instances of the first is much larger than that of the latter.

Index Terms—CF-integrals, d-Choquet integrals, Choquet integral, fuzzy rule based classification systems, pre-aggregation functions

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I. INTRODUCTION

An aggregation function [1] is a special type of function that fuses different values into a single one, which represents all the considered values. The simple mean, the Ordered Weight Average (OWA) [2] and the Choquet integral (CI) [3] are examples of aggregation operators. Aggregation functions have an important role in Fuzzy Rule-Based Classification Systems (FRBCSs) [4], since they are responsible for aggregating the information related with the different classes of the problem in the Fuzzy Reasoning Method (FRM) [5]. They are also applied in multi-criteria decision making (MCDM) [6].

While the FRM of Winning Rule (WR) [7] takes into account only the values having the largest compatibility with the problem, the usage of the CI in the FRM allows to model the relation among these values by considering the difference of the elements in relation to a fuzzy measure [8]. In fact, Barrenechea et al. [9] introduced a FRM considering the CI, and obtained an improvement in the performance of the classifier when associated to the power measure.

The CI was generalized in many ways and used in the FRM of FRBCSs, generalizations such as the C_T -integrals [10] (also applied in MCDM [11]), CC-integrals [12] (also applied in motor-imagery-based brain computer interface systems [13]), C_F -integrals [14] and $C_{F_1F_2}$ -integrals [15] introduced by Luca et al., and $gC_{F_1F_2}$ -integrals by Dimuro et al. [16]. For a comprehensive analysis of generalizations of the CI see [17].

Concerning the main features of those generalizations, observe that some of them may neither be aggregation functions (since they may not be increasing in its standard sense) nor averaging (when the output of the “aggregation” operator is bounded by the minimum and the maximum of the inputs). Table I shows an overview of such characteristics, which depend on specific properties of the functions used in the generalization, where T is a t-norm [18], C is a copula [19] and F , F_1 and F_2 are more general functions.

Recently, by replacing the difference operator in the definition of the standard CI by restricted dissimilarity functions [23], [24], Bustince et al. [25] introduced the concept of d-Choquet integrals. This interesting generalization can

TABLE I: Main features of the generalizations of the CI

Integral	Incr. (AF)	D. Incr. (PAF)	OD incr. -	Aver.	Non-aver.
CI	✓			✓	
C_T^*		✓		✓	
CC	✓			✓	
C_F		✓		✓	✓
$C_{F_1 F_2}^{**}$			✓	✓	✓
$gC_{F_1 F_2}$	✓	✓	✓	✓	✓

* When T is different from the product t-norm; ** when F_1 and F_2 are not copulae.
 Incr.: increasing; D. Incr.: directional increasing [20]; OD incr.: ordered directional increasing [21].
 AF: aggregation function [1]; PAF: pre-aggregation function [10].
 Aver.: averaging [22]; Non-aver.: Non averaging [22].

improve the usability of CI, since it can be used when comparison between inputs using the usual difference is not possible/productive/viable, as in the case of intervals. Moreover, since there are several ways of defining such dissimilarity functions, one can adopt the one that best fit the faced problem.

Therefore, in an attempt to improve both the performance and usability of C_F -integrals in FRBCSs the general objective of this paper is to introduce the concept of dC_F -integrals, which is a generalization of the Choquet-based C_F -integrals by replacing the difference operator by restricted dissimilarity functions. For that, we have two specific goals: (i) a theoretical study, showing the main features of this new aggregation-like function according to the function F and restricted dissimilarity functions used in the construction; and (ii) the application of dC_F -integrals in a FRM of a FRBCS, performing an extensive analysis of its behaviour and performance. In this sense, we aim at answer the following research questions:

- 1) In which circumstances are dC_F -integrals able to improve the performance of the classifier, when compared with C_F -integrals?
- 2) Which combinations of functions F , restricted dissimilarity functions and fuzzy measures perform better?
- 3) Do dC_F -integrals enlarge the applicability of C_F -integrals?

In order to present a complete and robust study, we consider 33 different datasets selected from KEEL dataset repository [26]. We combine 21 different functions F with six different restricted dissimilarity functions. The functions F and restricted dissimilarity functions combinations are combined with five different fuzzy measures. The performance of the dC_F -integrals are measured using the accuracy rate and the results are supported and analyzed considering statistical tests.

The organization of this paper follows this structure. Section II presents the preliminaries concepts. In Section III, we introduce the concept of dC_F -integrals joint with a theoretical study. The new FRM is presented in Section IV. The experimental framework is described in Section V. After that, the obtained results are analysed in Section VI. Finally, the conclusions are drawn in Section VII.

II. PRELIMINARIES

A function $F : [0, 1]^2 \rightarrow [0, 1]$ with 0 as *left annihilator element* (0-LAE), that is, $F(0, y) = 0$, $\forall y \in [0, 1]$, is said to be left 0-absorbent. If $F(x, 1) = x$, for any $x \in [0, 1]$, then we say that it has 1 as *Right Neutral Element* (1-RNE). Also,

when $F(x, y) \leq x$, $\forall x, y \in [0, 1]$, we say that F follows the *Left Conjunctive Property* (LC). [14]

Since we are working with generalizations of the CI two definitions are essential. The first one is the definition of aggregation functions [1]. Then, let $A : [0, 1]^n \rightarrow [0, 1]$ be an n -ary function. If A satisfies:

- (A1) Increasingness in each argument: $\forall i \in \{1, \dots, n\}$: if $x_i \leq y$ then $A(x_1, \dots, x_n) \leq A(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n)$;
- (A2) Boundary conditions: $A(0, \dots, 0) = 0$, $A(1, \dots, 1) = 1$; then F is an *aggregation function* (AF).

The second is a more generic definition, where we ask F to be increasing only in a pre-defined direction, that is, to be *directional monotonic* [20]. Let F be a bivariate function and $\mathbf{r} = (r_1, \dots, r_n)$ an n -dimensional vector, with $\mathbf{r} \neq \mathbf{0}$. We say that F is *$\mathbf{r}$ -increasing* if, for all $\mathbf{x} \in [0, 1]^n$ and all $c > 0$ such that $(\mathbf{x} + c\mathbf{r}) \in [0, 1]^n$, the following inequality holds

$$F(x_1 + cr_1, \dots, x_n + cr_n) \geq F(x_1, \dots, x_n).$$

If F is $\mathbf{r}$ -increasing and satisfies the boundary conditions (A2), then F is an *$\mathbf{r}$ -pre-aggregation function* ($\mathbf{r}$ -PAF) [10], [27].

By working with fuzzy integrals we work directly with *fuzzy measures* [3], that is, a function $m : 2^N \rightarrow [0, 1]$ for which, for all $X, Y \subseteq N = \{1, \dots, n\}$, the following properties holds:

- (m1) Increasingness: if $X \subseteq Y$, then $m(X) \leq m(Y)$;
- (m2) Boundary conditions: $m(\emptyset) = 0$ and $m(N) = 1$.

Using a fuzzy measure $m : 2^N \rightarrow [0, 1]$, the discrete *Choquet integral* (CI) is defined $\mathfrak{C}_m : [0, 1]^n \rightarrow [0, 1]$ with respect to m , for all $\mathbf{x} \in [0, 1]^n$, by

$$\mathfrak{C}_m(\mathbf{x}) = \sum_{i=1}^n (x_{(i)} - x_{(i-1)}) \cdot m(A_{(i)}),$$

where $(x_{(1)}, \dots, x_{(n)})$ is an increasing permutation of $\mathbf{x}$, with $x_{(0)} = 0$ and $A_{(i)} = \{(i), \dots, (n)\}$ is the subset of indices of $n - i + 1$ largest components of $\mathbf{x}$.

As discussed in the Introduction, several generalizations of the CI may be found in the literature [17]. Recently, Lucca et al. [14] introduced the concept of C_F -integral. Let $F : [0, 1]^2 \rightarrow [0, 1]$ be a bivariate function. The C_F -integral with respect to a fuzzy measure $m : 2^N \rightarrow [0, 1]$ is the function $C_m^F : [0, 1]^n \rightarrow [0, 1]$ defined, for all $\mathbf{x} \in [0, 1]^n$, by

$$\mathfrak{C}_m^F(\mathbf{x}) = \min \left\{ 1, \sum_{i=1}^n F(x_{(i)} - x_{(i-1)}, m(A_{(i)})) \right\},$$

where $\mathbf{x}$ and $A_{(i)}$ were defined in the previous paragraph for the CI. For examples of functions F , see Table II.

As a key concept in this work, a *restricted dissimilarity function* [23], [24] is a function $\delta : [0, 1]^2 \rightarrow [0, 1]$ that satisfies, for all $x, y, z \in [0, 1]$, the following conditions:

- (d1) $\delta(x, y) = \delta(y, x)$;
- (d2) $\delta(x, y) = 1$ if and only if $\{x, y\} = \{0, 1\}$;
- (d3) $\delta(x, y) = 0$ if and only if $x = y$;
- (d4) if $x \leq y \leq z$, then $\delta(x, y) \leq \delta(x, z)$ and $\delta(y, z) \leq \delta(x, z)$.

By replacing the difference operator in the definition of the CI by a restricted dissimilarity function, Bustince et al.

[25] introduced the d-Choquet integral (d-integral, for short). A discrete *d-Choquet integral* with respect to a fuzzy measure $m : 2^N \rightarrow [0, 1]$ and a restricted dissimilarity function $\delta : [0, 1]^2 \rightarrow [0, 1]$ is a mapping $C_{m,\delta} : [0, 1]^n \rightarrow [0, 1]$, defined, for all $\mathbf{x} \in [0, 1]^n$, by:

$$C_{m,\delta}(\mathbf{x}) = \sum_{i=1}^n \delta(x_{(i)}, x_{(i-1)}) \cdot m(A_{(i)})$$

where $\mathbf{x}$ and $A_{(i)}$ were defined previously. For examples of restricted dissimilarity functions, see Table III (functions δ).

III. INTRODUCING dC_F -INTEGRALS

This section introduces the definition of dC_F -integral, analysing the most important properties.

Definition 1 (dC_F -integral). Let $F : [0, 1]^2 \rightarrow [0, 1]$ be a bivariate function, $\delta : [0, 1]^2 \rightarrow [0, 1]$ be a restricted dissimilarity function and $m : 2^N \rightarrow [0, 1]$ be a fuzzy measure. Then, the generalization of the CI by the function F , with respect to δ and m , called dC_F -integral, is the function $\mathfrak{C}_{F,m,\delta} : [0, 1]^n \rightarrow [0, 1]$, defined, for all $\mathbf{x} \in [0, 1]^n$, by:

$$\mathfrak{C}_{F,m,\delta}(\mathbf{x}) = \min \left\{ 1, \sum_{i=1}^n F(\delta(x_{(i)}, x_{(i-1)}), m(A_{(i)})) \right\}$$

where $(x_{(1)}, \dots, x_{(n)})$ is an increasing permutation on the input $\mathbf{x}$, with $x_{(0)} = 0$ and $A_{(i)} = \{(i), \dots, (n)\}$.

It can be noticed that the definition above is well-defined for any $x_{(i)} \in [0, 1]$, $\delta : [0, 1]^2 \rightarrow [0, 1]$ and $F : [0, 1]^2 \rightarrow [0, 1]$.

In the following results, consider all fuzzy measures $m : 2^N \rightarrow [0, 1]$, functions $F : [0, 1]^2 \rightarrow [0, 1]$ and restricted dissimilarity functions $\delta : [0, 1]^2 \rightarrow [0, 1]$.

Theorem 1 (Boundary Conditions). (i) $\mathfrak{C}_{F,m,\delta}$ satisfies the 0-boundary condition if and only if F satisfies (0-LAE).

(ii) If F satisfies (0-LAE) then $\mathfrak{C}_{F,m,\delta}$ satisfies the 1-boundary condition if and only if $F(1, 1) \geq 1$.

(iii) If $F(1, 1) \geq 1$ then $\mathfrak{C}_{F,m,\delta}$ satisfies the 1-boundary condition.

Proof. (i) ($\Leftarrow$) Take $\mathbf{x} = \mathbf{0} = (0, \dots, 0)$. Then:

$$\begin{aligned} \mathfrak{C}_{F,m,\delta}(\mathbf{0}) &= \min \left\{ 1, \sum_{i=1}^n F(\delta(0, 0), m(A_{(i)})) \right\} \\ &= \min \left\{ 1, \sum_{i=1}^n F(0, m(A_{(i)})) \right\} \quad \text{by (d3)} \\ &= 0 \quad \text{(by 0-LAE)} \end{aligned}$$

($\Rightarrow$) Suppose that $\mathfrak{C}_{F,m,\delta}(\mathbf{0}) = 0$. Then, similar to the proof above, one has that $\sum_{i=1}^n F(0, m(A_{(i)})) = 0$, and, then, F satisfies (0-LAE).

(ii) Suppose that F satisfies (0-LAE). ($\Leftarrow$) For $\mathbf{x} = \mathbf{1} = (1, \dots, 1)$, since $F(1, 1) \geq 1$, we have

$$\begin{aligned} \mathfrak{C}_{F,m,\delta}(\mathbf{x}) &= \min \left\{ 1, F(\delta(1, 0), 1) + \sum_{i=2}^n F(\delta(1, 1), m(A_{(i)})) \right\} \end{aligned}$$

$$\begin{aligned} &= \min \left\{ 1, F(1, 1) + \sum_{i=2}^n F(0, m(A_{(i)})) \right\} \quad \text{by (d2), (d3)} \\ &= \min \left\{ 1, 1 + 0 \right\} = 1 \quad \text{by (0-LAE)} \end{aligned}$$

($\Rightarrow$) Suppose that $\mathfrak{C}_{F,m,\delta}(\mathbf{1}) = 1$. Then, similar to the proof above, one has that $\min\{1, F(1, 1) + \sum_{i=2}^n F(0, m(A_{(i)}))\} = 1$, and then, $F(1, 1) \geq 1$.

(iii) Analogous to the proof of (ii)($\Leftarrow$). $\square$

Theorem 2 (Directional monotonicity). If F is (1, 0)-increasing and, for all $a, b \in [0, 1]$, $h > 0$, with $a + h, b + h \in [0, 1]$, it holds that:

$$\delta(a + h, b + h) \geq \delta(a, b), \quad (1)$$

then $\mathfrak{C}_{F,m,\delta}$ is 1-increasing

Proof. Consider F to be (1, 0)-increasing. For any $\mathbf{x} \in [0, 1]^n$, $c > 0$, with $\mathbf{x} + c \in [0, 1]^n$, consider that (1) holds whenever $h = c$, $a = x_{(i)}$ and $b = x_{(i-1)}$, for any $i = 2, \dots, n$, that is, $\delta(x_{(i)} + c, x_{(i-1)} + c) \geq \delta(x_{(i)}, x_{(i-1)})$. It follows that:

$$\begin{aligned} \mathfrak{C}_{F,m,\delta}(\mathbf{x} + c) &= \min \left\{ 1, F(\delta(x_{(1)} + c, 0), 1) \right. \\ &\quad \left. + \sum_{i=2}^n F(\delta(x_{(i)} + c, x_{(i-1)} + c), m(A_{(i)})) \right\} \\ &> \min \left\{ 1, F(\delta(x_{(1)}, 0), 1) \right. \\ &\quad \left. + \sum_{i=1}^n F(\delta(x_{(i)}, x_{(i-1)}), m(A_{(i)})) \right\} \\ &\quad \text{by (d3), (d4), (1) and (1, 0)-increasingness of } F \\ &= \mathfrak{C}_{F,m,\delta}(\mathbf{x}). \end{aligned}$$

Therefore, $\mathfrak{C}_{F,m,\delta}$ is 1-increasing. $\square$

From theorems 1 and 2, it follows that:

Theorem 3 (PAF). If F is (1, 0)-increasing, satisfies (0-LAE), $F(1, 1) \geq 1$ and condition (1) holds then $\mathfrak{C}_{F,m,\delta}$ is an 1-PAF.

Theorem 4 (Monotonicity). $\mathfrak{C}_{F,m,\delta}$ is increasing if and only if the following conditions hold:

(i) F is (1, 0)-increasing;

(ii) For all $z_1, z_2, z_3, z_4 \in [0, 1]$, with $z_1 \leq z_2 \leq z_3 \leq z_4$:

$$\delta(z_1, z_3) + \delta(z_3, z_4) \geq \delta(z_1, z_2) + \delta(z_2, z_4); \quad (2)$$

(iii) For all $a_1, a_2, b_1, b_2 \in [0, 1]$, $h_1, h_2 \geq 0$, with $a_1 + h_1, a_2 + h_2 \in [0, 1]$: If $b_2 \leq b_1$ and $h_1 \geq h_2$ then:

$$F(a_1 + h_1, b_1) - F(a_2 + h_2, b_2) \geq F(a_1, b_1) - F(a_2, b_2). \quad (3)$$

Proof. ($\Leftarrow$) Take $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ such that, for $k \in \{1, \dots, n\}$ and $\lambda > 0$, it holds that $x_{(k)} = y_{(k)} + \lambda \in [0, 1]$, and, for all $i \neq k$, $x_{(i)} = y_{(i)}$. If $\sum_{i=1}^n F(\delta(x_{(i)}, x_{(i-1)}), m(A_{(i)})) \geq 1$, then it is immediate that $\mathfrak{C}_{F,m,\delta}(\mathbf{x}) \geq \mathfrak{C}_{F,m,\delta}(\mathbf{y})$. Otherwise, one has the following possibilities:

(a) $k = n$: In this case, $x_{(n)} = y_{(n)} + \lambda$ and, since F is $(1, 0)$ -increasing, it follows that:

$$\begin{aligned} \mathfrak{C}_{F,m,\delta}(\mathbf{x} = \min & \left\{ 1, \sum_{i=1}^{n-1} F(\delta(y_{(i)}, y_{(i-1)}), m(A_{(i)})) \right. \\ & \left. + F(\delta(y_{(n)} + \lambda, y_{(n-1)}), m(A_{(n)})) \right\} \\ & \geq \min \left\{ 1, \sum_{i=1}^{n-1} F(\delta(y_{(i)}, y_{(i-1)}), m(A_{(i)})) \right. \\ & \left. + F(\delta(y_{(n)}, y_{(n-1)}), m(A_{(n)})) \right\} \text{ by (d4)} \\ & = \mathfrak{C}_{F,m,\delta}(\mathbf{y}). \end{aligned}$$

(b) $1 \leq k < n$: First, observe that

$$x_{(k-1)} = y_{(k-1)} \leq y_{(k)} < x_{(k)} = y_{(k)} + \lambda \leq x_{(k+1)} = y_{(k+1)}.$$

By (d3) and (d4), one has that

$$\delta(y_{(k)} + \lambda, y_{(k-1)}) > \delta(y_{(k)}, y_{(k-1)}) \quad (4)$$

$$\delta(y_{(k+1)}, y_{(k)} + \lambda) < \delta(y_{(k+1)}, y_{(k)}) \quad (5)$$

For simplicity denote $\delta(y_{(k)}, y_{(k-1)}) = a_1$, $\delta(y_{(k)} + \lambda, y_{(k-1)}) = a_1 + h_1$, $\delta(y_{(k+1)}, y_{(k)} + \lambda) = a_2$ and $\delta(y_{(k+1)}, y_{(k)}) = a_2 + h_2$, where $h_1 = \delta(y_{(k)} + \lambda, y_{(k-1)}) - \delta(y_{(k)}, y_{(k-1)}) > 0$ and $h_2 = \delta(y_{(k+1)}, y_{(k)}) - \delta(y_{(k+1)}, y_{(k)} + \lambda) > 0$. Notice that $h_1 \geq h_2$, since condition (2) holds. Also denote $b_1 = m(A_{(k)})$ and $b_2 = m(A_{(k+1)})$ and notice that $b_2 \leq b_1$. Then it follows that:

$$\begin{aligned} \mathfrak{C}_{F,m,\delta}(\mathbf{x}) &= \min \left\{ 1, \sum_{i=1}^{k-1} F(\delta(y_{(i)}, y_{(i-1)}), m(A_{(i)})) \right. \\ & \quad + F(\delta(y_{(k)} + \lambda, y_{(k-1)}), m(A_{(k)})) \\ & \quad + F(\delta(y_{(k+1)}, y_{(k)} + \lambda), m(A_{(k+1)})) \\ & \quad \left. + \sum_{i=k+2}^n F(\delta(y_{(i)}, y_{(i-1)}), m(A_{(i)})) \right\} \\ &= \min \left\{ 1, \sum_{i=1}^{k-1} F(\delta(y_{(i)}, y_{(i-1)}), m(A_{(i)})) \right. \\ & \quad + F(a_1 + h_1, b_1) + F(a_2, b_2) \\ & \quad \left. + \sum_{i=k+2}^n F(\delta(y_{(i)}, y_{(i-1)}), m(A_{(i)})) \right\} \\ &\geq \min \left\{ 1, \sum_{i=1}^{k-1} F(\delta(y_{(i)}, y_{(i-1)}), m(A_{(i)})) \right. \\ & \quad + F(a_1, b_1) + F(a_2 + h_2, b_2) \\ & \quad \left. + \sum_{i=k+2}^n F(\delta(y_{(i)}, y_{(i-1)}), m(A_{(i)})) \right\} \text{ by (4), (5), (3)} \\ &= \min \left\{ 1, \sum_{i=1}^{k-1} F(\delta(y_{(i)}, y_{(i-1)}), m(A_{(i)})) \right. \\ & \quad \left. + F(\delta(y_{(k)}, y_{(k-1)}), m(A_{(k)})) \right\} \end{aligned}$$

$$\begin{aligned} & + F(\delta(y_{(k+1)}, y_{(k)}), m(A_{(k+1)})) \\ & + \sum_{i=k+2}^n F(\delta(y_{(i)}, y_{(i-1)}), m(A_{(i)})) \Big\} = \mathfrak{C}_{F,m,\delta}(\mathbf{y}). \end{aligned}$$

($\Rightarrow$) Since $\mathfrak{C}_{F,m,\delta}$ is monotonic, then for all $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ such that there is $k \in \{1, \dots, n\}$ and $\lambda > 0$ for which $x_{(k)} = y_{(k)} + \lambda \in [0, 1]$, and for any $i \in \{1, \dots, n\}$ with $i \neq k$, we have $x_{(k)} = y_{(k)}$, it holds that:

$$\begin{aligned} \mathfrak{C}_{F,m,\delta}(\mathbf{x}) - \mathfrak{C}_{F,m,\delta}(\mathbf{y}) &\geq 0 \\ \Leftrightarrow \sum_{i=1}^n F(\delta(x_{(i)}, x_{(i-1)}), m(A_{(i)})) & \\ - \sum_{i=1}^n F(\delta(y_{(i)}, y_{(i-1)}), m(A_{(i)})) &\geq 0 \\ \Leftrightarrow \sum_{i=1}^n \left(F(\delta(x_{(i)}, x_{(i-1)}), m(A_{(i)})) \right. & \\ \left. - F(\delta(y_{(i)}, y_{(i-1)}), m(A_{(i)})) \right) &\geq 0 \end{aligned}$$

Here, the only non-zero elements of the sum are the ones that contain the k -th element:

$$\begin{aligned} & F(\delta(x_{(k)}, x_{(k-1)}), m(A_{(k)})) \\ & - F(\delta(y_{(k)}, y_{(k-1)}), m(A_{(k)})) \\ & + F(\delta(x_{(k+1)}, x_{(k)}), m(A_{(k+1)})) \\ & - F(\delta(y_{(k+1)}, y_{(k)}), m(A_{(k+1)})) \geq 0 \\ \Leftrightarrow & F(\delta(y_{(k)} + \lambda, y_{(k-1)}), m(A_{(k)})) \\ & - F(\delta(y_{(k)}, y_{(k-1)}), m(A_{(k)})) \\ & + F(\delta(y_{(k+1)}, y_{(k)} + \lambda), m(A_{(k+1)})) \\ & - F(\delta(y_{(k+1)}, y_{(k)}), m(A_{(k+1)})) \geq 0 \quad (6) \end{aligned}$$

Then, we have the following possibilities:

(a) $k = n$: In this case $x_{(n)} = y_{(n)} + \lambda$ and

$$\begin{aligned} & F(\delta(y_{(n)} + \lambda, y_{(n-1)}), m(A_{(n)})) \\ & - F(\delta(y_{(n)}, y_{(n-1)}), m(A_{(n)})) \geq 0, \end{aligned}$$

and, therefore, by (d4), F is $(1, 0)$ -increasing.

(b) $1 \leq k < n$: by using the previous notation adopted in the item (a) of the proof of ($\Leftarrow$), we have that $b_2 \leq b_1$. Denoting $z_1 = y_{(k-1)}$, $z_2 = y_{(k)}$, $z_3 = y_{(k)} + \lambda$ and $z_4 = y_{(k+1)}$, we have that $z_1 \leq z_2 \leq z_3 \leq z_4$ and whenever $h_1 \geq h_2$, we have that condition (2) holds. Finally, Eq. (6) can be rewritten as:

$$\begin{aligned} & F(a_1 + h_1, b_1) - F(a_1, b_1) + F(a_2, b_2) - F(a_2 + h_2, b_2) \geq 0 \\ \Leftrightarrow & F(a_1 + h_1, b_1) - F(a_2 + h_2, b_2) \geq F(a_1, b_1) - F(a_2, b_2) \end{aligned}$$

and, then, condition (3) also holds. $\square$

From theorems 1 and 4, it follows that:

Theorem 5 (AF). $\mathfrak{C}_{F,m,\delta}$ is an aggregation function if and only if F is $(1, 0)$ -increasing, satisfies (0-LAE), $F(1, 1) \geq 1$ and conditions (2) and (3) of Theorem 4 hold.

Proposition 1. If F satisfies (1-RNE) and (0-LAE), and $\delta(x, 0) = x$, then $\mathfrak{C}_{F,m,\delta}$ is idempotent.

TABLE II: (1,0)-increasing functions F used in this study

Definition	Description
$T_M(x, y) = \min\{x, y\}$	Minimum t-norm
$T_P(x, y) = xy$	Algebraic product
$T_L(x, y) = \max\{0, x + y - 1\}$	Łukasiewicz
$T_{DP}(x, y) = \begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$	Drastic Product
$T_{NM}(x, y) = \begin{cases} \min\{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$	Nilpotent Minimum
$T_{HP}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$	Hamacher Product
$O_B(x, y) = \min\{x\sqrt{y}, y\sqrt{x}\}$	[28], Cuadras-Augé copula [29]
$O_{mM}(x, y) = \min\{x, y\} \max\{x^2, y^2\}$	[30], [31], [32]
$O_\alpha(x, y) = xy(1 + \alpha(1 - x)(1 - y))$, with $\alpha \in [-1, 1] \setminus \{0\}$	[19], Farlie-Gumbel-Morgenstern copula family
$O_{Div}(x, y) = \frac{xy + \min\{x, y\}}{2}$	[19], [12]
$GM(x, y) = \sqrt{xy}$	Geometric Mean, [33]
$HM(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \\ \frac{2}{\frac{1}{x} + \frac{1}{y}} & \text{otherwise.} \end{cases}$	Harmonic Mean, [33]
$Sin(x, y) = \sin\left(\frac{\pi}{2}(xy)^{\frac{1}{4}}\right)$	Sine, [33]
$O_{RS}(x, y) = \min\left\{\frac{(x+1)\sqrt{y}}{2}, y\sqrt{x}\right\}$	
$C_F(x, y) = xy + x^2y(1-x)(1-y)$	[18], [12]
$C_L(x, y) = \max\{\min\{x, \frac{y}{2}\}, x + y - 1\}$	[19], [12]
$F_{GL}(x, y) = \sqrt{\frac{x(y+1)}{2}}$	
$F_{BPC}(x, y) = xy^2$	[34]
$F_{BD1}(x, y) = \min\{x, 1 - x + \min\{x, y^q\}\}$, with $0 < q \leq 1$	[14], [15]
$F_{NA}(x, y) = \begin{cases} x & \text{if } x \leq y \\ \min\{\frac{x}{2}, y\} & \text{otherwise.} \end{cases}$	[14], [15]
$F_{NA2}(x, y) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{x+y}{2} & \text{if } 0 < x \leq y \\ \min\{\frac{x}{2}, y\} & \text{otherwise.} \end{cases}$	[14], [15]

Proof. Consider $\mathbf{x} = \{x, \dots, x\} \in [0, 1]^n$. Then:

$$\begin{aligned}
& \mathfrak{C}_{F,m,\delta}(\mathbf{x}) \\
&= \min \left\{ 1, F(\delta(x, 0), 1) + \sum_{i=2}^n F(\delta(x, x), m(A_{(i)})) \right\} \\
&= \min \left\{ 1, \delta(x, 0) + \sum_{i=2}^n F(0, m(A_{(i)})) \right\} \text{ by (d3), (1-RNE)} \\
&= \min \{1, \delta(x, 0)\} \text{ by (0-LAE)} \\
&= \delta(x, 0) = x.
\end{aligned}$$

Therefore, $\mathfrak{C}_{F,m,\delta}(\mathbf{x})$ is idempotent. $\square$

Proposition 2. *If F satisfies (1-RNE) and $\delta(0, x) \geq x$, then $\mathfrak{C}_{F,m,\delta}(\mathbf{x}) \geq \min(\mathbf{x})$, for all $\mathbf{x} \in [0, 1]^n$.*

Proof. Consider $\mathbf{x} \in [0, 1]^n$. Then:

$$\begin{aligned}
& \mathfrak{C}_{F,m,\delta}(\mathbf{x}) \\
&= \min \left\{ 1, F(\delta(x_{(1)}, 0), 1) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=2}^n F(\delta(x_{(i)}, x_{(i-1)}), m(A_{(i)})) \Big\} \\
&= \min \left\{ 1, \delta(x_{(1)}, 0) + \sum_{i=2}^n F(\delta(x_{(i)}, x_{(i-1)}), m(A_{(i)})) \right\} \\
& \hspace{15em} \text{by (1-RNE)} \\
&\geq \min\{1, \delta(x_{(1)}, 0)\} = \delta(x_{(1)}, 0) \geq x_{(1)} = \min(\mathbf{x}).
\end{aligned}$$

Therefore, $\mathfrak{C}_{F,m,\delta}(\mathbf{x}) \geq \min(\mathbf{x})$. $\square$

Proposition 3. *If F satisfies (LC) and $\sum_{i=1}^n \delta(x_{(i)}, x_{(i-1)}) \leq x_{(n)}$, then $\mathfrak{C}_{F,m,\delta}(\mathbf{x}) \leq \max(\mathbf{x})$, for all $\mathbf{x} \in [0, 1]^n$.*

Proof. Consider $\mathbf{x} \in [0, 1]^n$. Then:

$$\begin{aligned}
\mathfrak{C}_{F,m,\delta}(\mathbf{x}) &= \min \left\{ 1, \sum_{i=1}^n F(\delta(x_{(i)}, x_{(i-1)}), m(A_{(i)})) \right\} \\
&\leq \min \left\{ 1, \sum_{i=1}^n \delta(x_{(i)}, x_{(i-1)}) \right\} \text{ by (LC)} \\
&\leq \min\{1, x_{(n)}\} = x_{(n)} = \max(\mathbf{x}).
\end{aligned}$$

Therefore, $\mathfrak{C}_{F,m,\delta}(\mathbf{x}) \leq \max(\mathbf{x})$. $\square$

Table III shows examples of combinations of functions F and δ that satisfy properties like aggregation (Agg.), 1-pre-aggregation (1-PAF), idempotency (Idem.) and averaging (Ave.). Notice that the only functions F that satisfies condition (3) are T_P and F_{BPC} . Also, condition (2) is satisfied by the restricted dissimilarity functions δ_0 , δ_3 and δ_4 . Observe that for the restricted dissimilarity function δ_3 one has that Theorem 2 for directional monotonicity is inconclusive, but whenever $\mathfrak{C}_{F,m,\delta_3}$ is an aggregation function it must be an 1-pre-aggregation function.

IV. dC_F -INTEGRALS IN THE FRM OF FRBCSS

In this section, we present the application of the dC_F -integral in the FRM. Considering a classification problem containing t training examples $\mathbf{x}_p = (x_{p1}, \dots, x_{pn}, y_p)$, with $p = 1, \dots, t$, where each x_{pi} is the value of the $i = 1, \dots, n$ variable, with $i = 1, \dots, n$, being the value of the i -th variable and $y_p \in \mathbb{C} = \{C_1, \dots, C_M\}$ is the label of the class of the p -th training example, and M is the number of classes.

In this paper, we focus on FRBCSSs. Specifically, we use the Fuzzy Association Rule-based Classification model for High Dimensional Problems (FARC-HD) [35]. The structure of the fuzzy rules generated by this classifier has the following form:

$$\begin{aligned}
& \text{Rule } R_j : \text{ If } x_1 \text{ is } A_{j1} \text{ and } \dots \text{ and } x_n \text{ is } A_{jn} \\
& \hspace{15em} \text{then Class is } C_j \text{ with } RW_j,
\end{aligned} \tag{7}$$

where R_j is the label of the j -th rule, A_{ji} is a fuzzy set representing a linguistic term modeled by a triangular shaped membership function. C_j is the class label and $RW_j \in [0, 1]$ is the rule weight [36], which in this case is computed as the confidence of the fuzzy rule.

Following the same approach used in the previous generalizations of the Choquet integral (See [10], [12], [14] and [15]), we modify the classical FRM of FARC-HD to include the dC_F -integrals in its third stage. Thus, the final

TABLE III: Properties of the dC_F -integral for various F and restricted dissimilarity functions, based on the results presented in this paper. (Here, m means that $\mathfrak{C}_{F,m,\delta}(x) \geq \min(x)$, M means that $\mathfrak{C}_{F,m,\delta}(x) \leq \max(x)$ and * means that our theorem is inconclusive in this case).

Function	$\delta_0 = x - y $				$\delta_1 = (x - y)^2$				$\delta_2 = \sqrt{ x - y }$				$\delta_3 = \sqrt{x} - \sqrt{y} $				$\delta_4 = x^2 - y^2 $				$\delta_5 = (\sqrt{x} - \sqrt{y})^2$			
	Agg.	1-PA	Idem.	Ave	Agg.	1-PA	Idem.	Ave	Agg.	1-PA	Idem.	Ave	Agg.	1-PA	Idem.	Ave	Agg.	1-PA	Idem.	Ave	Agg.	1-PA	Idem.	Ave
T_M			✓	✓				M				m				m				M			✓	✓
T_P	✓		✓	✓			✓	M			✓	m	✓	✓*		m	✓			✓			✓	✓
T_L			✓	✓			✓	M			✓	m				m	✓			✓			✓	✓
T_{DP}			✓	✓			✓	M			✓	m				m	✓			✓			✓	✓
T_{NM}			✓	✓			✓	M			✓	m				m	✓			✓			✓	✓
T_{HP}			✓	✓			✓	M			✓	m				m	✓			✓			✓	✓
O_B				M				M												M				M
O_{mM}			✓	✓			✓	M			✓	m				m	✓			✓			✓	✓
O_α			✓	✓			✓	M			✓	m				m	✓			✓			✓	✓
O_{Div}			✓	✓			✓	M			✓	m				m	✓			✓			✓	✓
GM																								
HM																								
Sin																								
O_{RS}																								
C_F			✓	✓			✓	M			✓	m				m	✓			✓			✓	✓
C_L			✓	✓			✓	M			✓	m				m	✓			✓			✓	✓
F_{GL}																								
F_{BPC}	✓		✓	✓			✓	M			✓	m	✓	✓*		m	✓	✓		✓			✓	✓
F_{BD1}			✓	✓			✓	M			✓	m				m	✓			✓			✓	✓
F_{NA}			✓	✓			✓	M			✓	m				m	✓			✓			✓	✓
F_{NA2}																								

classification of an example is computed by:

$$class = \arg \max_k (C_k^{\mathfrak{C}_{F,m,\delta}}(\mu_{A_k}(x) * RW_k > 0)) \text{ with } k = 1, \dots, M.$$

where $C_k^{\mathfrak{C}_{F,m,\delta}}$ is the dC_F -integral considered to aggregated the fired rules for each class, μ_{A_k} is the matching degree of the example x_p , given by the product t-norm T_P applied to the membership degrees of x_p , related to the antecedent of the k-th rule conjunction and RW_k is the rule weight.

V. EXPERIMENTAL FRAMEWORK

In this section we present the experimental framework used in the study. We start detailing the considered datasets, then the used fuzzy measures. After that, we show the configuration of the proposal and, finally, we discuss the statistical tests that are used to validate the proposal.

A. Datasets used in the study

This study is conducted taking into consideration 33 different datasets selected from KEEL dataset repository [26]. We highlight that these datasets are the same ones used in previous studies, such as C_F -integrals [14] and C_{F1F2} -integrals [15]. This allow a comparison with state-of-the-art approaches.

We summarize the datasets in Table IV. For each dataset, we present the corresponding identification (Id), the number of instances (#Inst), attributes (#Atts) and classes (#Class).

In order to achieve the best result and avoid overfitting, we applied a 5-fold cross-validation procedure. That is, we split the datasets into five partitions containing 20% of the examples each one. This process is repeated five times to complete all iterations, and in each of the 5 steps, one different partition is used for testing and the other four are used for training. The general performance of the model is measured according to each testing partition, based on the accuracy rate

TABLE IV: Summary of the datasets used in the study.

Id.	Dataset	#Inst.	#Atts.	#Class
App	Appendicitis	106	7	2
Bal	Balance	625	4	3
Ban	Banana	5,300	2	2
Bnd	Bands	365	19	2
Bup	Bupa	345	6	2
Cle	Cleveland	297	13	5
Con	Contraceptive	1,473	9	3
Eco	Ecoli	336	7	8
Gla	Glass	214	9	6
Hab	Haberman	306	3	2
Hay	Hayes-Roth	160	4	3
Ion	Ionosphere	351	33	2
Iri	Iris	150	4	3
Led	led7digit	500	7	10
Mag	Magic	1,902	10	2
New	Newthyroid	215	5	3
Pag	Pageblocks	5,472	10	5
Pen	Penbased	10,992	16	10
Pho	Phoneme	5,404	5	2
Pim	Pima	768	8	2
Rin	Ring	740	20	2
Sah	Saheart	462	9	2
Sat	Satimage	6,435	36	7
Seg	Segment	2,310	19	7
Shu	Shuttle	58,000	9	7
Son	Sonar	208	60	2
Spe	Spectfheart	267	44	2
Tit	Titanic	2,201	3	2
Two	Twonorm	740	20	2
Veh	Vehicle	846	18	4
Win	Wine	178	13	3
Wis	Wisconsin	683	11	2
Yea	Yeast	1,484	8	10

(the number of correctly classified examples divided by the total number of examples). At the end, after calculating each partition performance, we use the average result of the five testing partitions to generate the output of the algorithm.

B. Considered Fuzzy Measures

The fuzzy measures considered in this paper, are the same that were used in [9], and have their performance studied in [37]. In what follows, consider that $A \subseteq N$:

Cardinality of uniform measure:

$$m(A) = \frac{|A|}{n} \quad (8)$$

Dirac's measure: For a previously fixed $i \in N$,

$$m(A) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A. \end{cases} \quad (9)$$

Weighted mean (Wmean): Take an arbitrary vector of weights $(w_1, \dots, w_n) \in [0, 1]^n$ such that $\sum_{i=1}^n w_i = 1$ and consider the fuzzy measure: $m(\{1\}) = w_1, \dots, m(\{n\}) = w_n$. For $|A| > 1$, one defines:

$$m(A) = \sum_{i \in A} m(\{i\}) \quad (10)$$

Ordered Weighted Averaging (OWA): Assign the following values for the fuzzy measure: $m(\{i\}) = w_j$, with i being the j -th largest component to be aggregated, that is, it considers an OWA operator. For $|A| > 1$, on defines:

$$m(A) = \sum_{i \in A} m(\{i\}) \quad (11)$$

Power Measure (PM):

$$m(A) = \left(\frac{|A|}{n} \right)^q, \text{ with } q > 0. \quad (12)$$

Having into consideration the PM, we highlight that the value used by the q exponent is genetically learned in this study. This approach follows the idea introduced in [9] and is widely used by the different generalizations of the Choquet integral (See [10], [12], [14] and [15]).

C. Configuration of the proposal

The new FRM presented in this paper, considering the concept of dC_F -integrals developed in Section III, is applied in the Fuzzy Association Rule-Based Classification method for High-Dimensional problems (FARC-HD) [35] fuzzy classifier. The configurations used by the algorithms are the following: linguistic labels per variable (5), conjunction operator (Product t-norm T_P), rule weight (Confidence), minimum support (0.05), minimum confidence (0.8), depth of the search tree (3), number of fuzzy rules that cover each example (2), population size (50), gray codification (30 bits per gene), number of evaluations (20.000).

D. Statistical test for performance comparisons

In this paper we consider hypothesis validation techniques to present a statistical analysis of the obtained results [38], [39]. Since the validity conditions of parametric tests are not satisfied, we consider the usage of non-parametric tests [40].

To perform group comparisons, we have used the Aligned Friedman rank test [41]. This test present in a reverse raking (the lowest rank is considered the better one) the best method in comparison to the remained approaches. Additionally, we compute the pos-hoc Holm's test [42] to indicate when the approach achieving the lowest ranking (control method) reject the null hypothesis. To do so, we compute the Adjust P-Value (APV) to be able to direct compare the control method with a level of significance α to different methods. Finally, to perform pairwise comparisons, we performed the Wilcoxon test [43].

VI. OBTAINED RESULTS

In this section, we show the results achieved when different dC_F -integrals are applied to aggregate the information in the FRM. In order to ease the understanding of the results and to better explain it, we have divided this study into three different parts. In the first one, we present a performance comparison among the different dC_F -integrals considering five different fuzzy measures. After that, we focus our analysis on the cases where dC_F -integrals are able to overcome the results obtained by Choquet based functions (that makes usage of the difference operator). We finish this section discussing which generalizations be suited when different combination of functions F and fuzzy measures are used.

In order to make a complete and robust study, we have the combinations of 5 different restricted dissimilarity functions with 21 generalizations of the Choquet integral using 5 different fuzzy measures. All this combinations applied in 33 datasets. In other words, we have conducted 525 experiments per dataset. Thus, for summarizing the proposal, the following results are related with the highest obtained mean among different combinations.

We make available in a git repository (<https://github.com/Giancarlo-Lucca/dCF-integrals>) the individual analysis of each operator. The results are separated in folds, according to each restricted dissimilarity function. For each fold, we present the results considering the fuzzy measures. Each fuzzy measure is related with 21 different dC_F -integrals.

A. Performance Analysis

To state the effectiveness of dC_F -integrals, the performance analysis was conducted following three different paths. The first one, take into consideration the local results. That is, we compared the results related with only one fuzzy measure, the original function and its generalizations by the five different dC_F -integrals. The second path follows a global analysis. On other words, the results obtained by all combinations are compared. Finally, we make an observation, for each fuzzy measure and function F , in the cases that they achieve the highest accuracy mean.

The obtained results are summarized in Table V. The rows present the different functions F used for the generalizations. The columns represent the combination of fuzzy measures and different dC_F -integrals. The dC_F -integrals using δ_0 (the difference operator) are the original C_F -integrals for the same F , which is our **baseline**, and we refer to it in our analyses. On the other hand, using δ_0 and $F = T_P$, we have the original d-Choquet integral. The values of each cell represent the mean obtained in the 33 considered dataset.

Aiming at extracting the maximum information of the results and ease their comprehension, we highlight in **bold-face** and underline the largest and lowest accuracy mean respectively, when a local analysis is performed. Considering the global analysis, we highlight with $+$ the method that achieves largest accuracy. Also, per column, the function F that presents the largest performance is marked with $*$.

Considering the results obtained in Table V and in a local analysis of the results, its noticeable that the dC_F -integral

TABLE V: Accuracy mean obtained by the functions F considered in the study combined with different fuzzy measures and dC_F -integrals

	Cardinality						Dirac						OWA					
	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5
T_M	79.41	77.69	79.99	79.61	78.57	80.38	79.41	77.44	78.82	78.41	77.78	79.30	79.17	77.44	79.35	78.97	77.98	79.46
T_P	79.02	77.84	80.17	78.90	78.02	80.56⁺*	79.02	77.44	78.82	78.41	77.78	79.30	78.64	76.76	79.49	78.42	77.19	79.20
T_L	77.12	77.09	77.17	76.75	77.24	77.95	77.12	77.44	78.82	78.41	77.78	79.30	77.22	76.85	77.17	76.79	77.05	77.00
T_{DP}	77.19	77.19	77.19	77.19	77.19	77.19	77.19	77.44	78.82	78.41	77.78	79.30⁺	77.19	77.19	77.19	77.19	77.19	77.19
T_{NM}	77.21	77.08	77.72	77.19	76.97	78.27	77.21	77.44	78.82	78.41	77.78	79.30	77.23	76.94	76.89	76.97	77.14	77.02
T_{HP}	79.41	77.70	80.16	79.71	78.36	80.26	79.41	77.44	78.82	78.41	77.78	79.30	79.40	76.97	79.82	79.02	77.90	79.53
O_B	79.05	77.59	80.08	79.26	78.17	79.97	79.05	77.44	78.82	78.41	77.78	79.30	78.89	76.61	79.55	79.06	77.37	79.58
O_{mM}	78.23	77.15	79.76	78.31	77.47	79.95	78.23	77.44	78.82	78.41	77.78	79.30	77.97	77.12	78.60	77.63	76.76	78.63
O_α	78.80	77.55	80.40⁺*	78.99	77.91	80.27	78.80	77.44	78.82	78.41	77.78	79.30	78.51	76.68	79.36	78.41	77.09	79.15
O_{div}	78.97	77.44	79.98	79.36	77.87	80.14	78.97	77.44	78.82	78.41	77.78	79.30	79.27	77.05	79.26	78.74	77.96	79.48
GM	80.33 [*]	79.13	80.14	80.40	79.70	80.00	80.33[*]	78.27	79.67	79.30	78.75	79.69	80.32	78.46	80.04	79.61	78.96	79.84
HM	79.64	78.00	80.28	79.75	79.42	80.05	79.64	77.52	79.24	78.59	77.86	79.07	79.53	77.12	80.05	79.29	78.26	79.72
Sin	80.12	80.12[*]	80.12	80.12	80.12	80.12	80.12	80.12[*]	80.12	80.12	80.12	80.12[*]	80.12	80.12[*]	80.12	80.12	80.12	80.12[*]
OR_S	80.17	79.02	80.33	80.56⁺*	79.26	80.24	80.17	78.27	79.67	79.30	78.75	79.69	80.12	77.69	79.72	79.20	78.83	79.66
C_F	78.52	77.46	79.88	78.98	77.92	80.24⁺	78.52	77.44	78.82	78.41	77.78	79.30	78.55	76.79	79.35	78.31	77.26	79.29
C_L	79.41	77.69	80.13	79.18	78.53	79.83	79.41	77.44	78.82	78.41	77.78	79.30	79.33	76.83	79.53	78.77	77.93	79.42
F_{GL}	80.15	79.26	80.26	79.91	80.21	80.43	80.15	79.12	80.40 [*]	80.61[*]	80.19 [*]	80.02	80.50[*]	79.42	80.40 [*]	80.33 [*]	80.20 [*]	79.99
F_{BPC}	77.72	77.24	79.30	78.13	77.44	79.72	77.72	77.44	78.82	78.41	77.78	79.30	78.09	77.41	78.74	77.65	77.13	78.57
F_{BD1}	79.60	77.89	80.29	79.64	78.67	79.96	79.60	77.51	80.13	79.40	78.60	80.00	79.52	78.04	79.88	79.22	78.35	79.99
F_{NA}	79.10	77.74	80.38⁺	79.24	78.39	79.84	79.10	77.44	78.82	78.41	77.78	79.30	79.19	77.07	79.89	79.00	77.97	79.79
F_{NA2}	80.16	80.11	79.98	80.15	80.34[*]	79.91	80.16	79.96	80.16	80.02	79.89	79.83	80.21	79.23	79.64	79.48	79.31	79.61
Mean	79.02	78.00	79.70	79.11	78.47	79.78	79.02	77.85	79.18	78.81	78.24	79.46	79.00	77.51	79.24	78.68	78.00	79.15

	Wmean						PM					
	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5
T_M	78.64	77.99	80.17	79.54	78.38	79.90	79.30	77.73	80.40	79.42	78.54	80.57⁺*
T_P	77.69	77.41	80.12	78.94	77.84	79.97	79.20	78.06	80.46	79.55	78.49	80.10
T_L	76.86	76.97	77.70	77.51	77.11	77.45	78.35	77.10	79.32	78.43	78.07	79.65⁺
T_{DP}	77.19	77.19	77.19	77.19	77.19	77.19	77.19	77.19	77.19	77.19	77.19	77.19
T_{NM}	77.12	76.88	77.77	77.39	76.92	78.16	79.02	77.21	79.83⁺	78.31	77.81	79.76
T_{HP}	78.71	77.66	79.77	79.47	78.60	80.26	79.74	77.76	80.26	79.47	78.53	80.33⁺
O_B	78.42	77.62	80.07	79.20	78.17	80.43⁺	79.44	77.74	80.49⁺	79.61	78.71	79.98
O_{mM}	77.14	77.39	79.49	78.12	77.53	79.71	79.19	77.84	80.06⁺	78.99	78.64	80.05
O_α	77.86	77.64	79.83	79.24	78.32	80.07	79.25	77.72	80.16	79.87	78.64	80.19
O_{div}	78.65	77.64	79.70	79.16	78.15	79.89	79.26	77.77	80.34⁺	79.53	78.61	80.24
GM	79.90	79.01	80.16	80.55⁺*	80.22	80.33	80.23	79.22	80.43	80.17	80.02	80.21
HM	79.37	78.37	80.16	79.83	78.92	79.78	80.28	78.32	80.30	79.82	79.06	80.36⁺
Sin	80.12	80.12	80.12	80.12	80.12	80.12	80.12	80.12	80.12	80.12	80.12	80.12
OR_S	79.49	78.64	80.31	79.97	79.26	80.15	80.46	79.20	80.30	80.10	80.19	80.23
C_F	77.75	77.57	80.03	79.06	77.73	79.78	79.34	77.79	80.05	79.52	78.47	80.23
C_L	78.47	77.64	79.84	79.22	78.45	79.99	79.25	77.56	80.11	79.74	78.68	80.41⁺
F_{GL}	80.32[*]	79.17	80.24	80.13	80.23 [*]	79.89	80.26	79.11	80.50[*]	80.15	80.39 [*]	80.34
F_{BPC}	77.10	77.39	79.39	78.12	77.58	79.45	79.19	77.87	80.25⁺	79.14	78.21	80.00
F_{BD1}	79.19	77.63	80.32[*]	79.80	78.59	79.91	79.79	77.67	79.98	79.41	78.61	80.43⁺
F_{NA}	78.66	77.83	79.93	79.41	78.61	79.94	79.64	77.61	80.27	79.43	78.91	79.91
F_{NA2}	80.03	80.40[*]	79.90	79.86	80.05	79.84	80.55⁺*	80.36 [*]	79.90	80.36 [*]	80.38	79.96
Mean	78.51	78.01	79.63	79.13	78.47	79.63	79.48	78.14	80.03	79.44	78.87	80.01

based on the dissimilarity function δ_1 achieved the lowest results considering all fuzzy measures. On the other hand, when δ_5 is used, the combination with the Cardinality, Dirac and Wmean showed an improvement of the performance.

The dC_F -integral based on the dissimilarity function δ_2 also present good results in a local analysis. In fact, this approach achieved the largest results when OWA, Wmean and PM are considered. Observe that using Wmean, δ_2 and δ_5 achieved the same general mean, equals to 79.63%, but δ_5 achieved the largest number of high accuracy scores in 9 different functions while δ_2 performed in 7. Notice that the mean of these approaches are very similar considering the measure PM.

The results obtained by the baseline approach δ_0 (considering the difference) and the dissimilarity function δ_3 presented a stable performance. This means that the obtained results, in general, are considered neither inferior nor superior.

By performing a global analysis, it is noticeable that the PM is the fuzzy measure achieving the largest accuracy mean. Precisely, the usage of this fuzzy measure combined with the 11 considered functions F and different dC_F -integrals led to a highest absolute accuracy mean, indicating the superiority of the method. F_{NA2} is the unique function that its baseline approach obtained a global result. The Cardinality also presented interesting results. When this fuzzy measure was taken into

TABLE VI: Summary of the functions that achieved the best performance per generalization and fuzzy measure.

	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5
Cardinality	GM	Sin	O_α	OR_S	F_{NA2}	T_P
Dirac	GM	Sin	F_{GL}	F_{GL}	F_{GL}	Sin
OWA	F_{GL}	Sin	F_{GL}	F_{GL}	F_{GL}	Sin
Wmean	F_{GL}	F_{NA2}	F_{BD1}	GM	F_{GL}	O_B
PM	F_{NA2}	F_{NA2}	F_{GL}	F_{NA2}	F_{GL}	T_M

consideration, the usage of 5 functions F improved the global performance. The remainder fuzzy measures did not perform so good as the previous ones. In fact, Dirac presented good results when combined with T_{DP} and F_{GL} , while Wmean behaved good only with GM . Observe that OWA did not achieve any superior performance, thus the combination with restricted dissimilarity functions could not be a good option.

The last analysis is focused on the functions F . Precisely, the aim is to show which function F per fuzzy measure achieved the highest accuracy mean among the 21 considered methods. We summarize this information and analysis in Table VI. The rows are related with the different fuzzy measures used in this study, the columns are the considered dC_F -integrals, and each cell contains the function F that achieved the highest accuracy mean.

Analyzing this table its noticeable that there are some

functions that are stable and fit well with different restricted dissimilarity functions and fuzzy measures. Precisely F_{GL} appeared 11 times, F_{NA2} and Sin 5 times, GM 3 times and the remainder appeared once. Another interesting point, is that the fuzzy measures that presented superior performance, Cardinality and PM, are the ones presenting the highest variability of functions. The functions F_{NA2} and F_{GL} appeared 4 and 2 times respectively, but all the other cases are different functions. In contrast, we can notice a dominance of F_{GL} and Sin when the other fuzzy measures are considered.

B. Comparing of dC_F -integrals with C_F -integrals

In this subsection we analyze if the usage of restricted dissimilarity functions are able to improve the performance of the baseline approach, namely, the C_F -integrals, considering different fuzzy measures.

Our analysis is supported by the Align Friedman rank test and the Holm's post hoc test. Precisely, for each fuzzy measure and dC_F -integrals, we applied these test in order to know which cases are considered as control variable and where there are statistical differences when compared to other approaches.

The obtained results are available in Table VII. It consist of the 21 considered function comparisons, considering the PM as fuzzy measure. Moreover, this table presents per function the considered dC_F -integrals, the obtained ranks and the APV. In the rows, where we represent the restricted dissimilarity measures, δ_0 was replaced by the respective C_F -integral notation (the baseline approach) for the sake of comparison. The methods are sorted from the lowest to the highest obtained rank where the lowest one is considered as control variable and it is compared with the remaining methods. The APV column indicates if there are statistical differences between the considered method and the control variable. When the obtained APV is inferior that 0.10% we highlight it with an underline, indicating that there is a statistical difference.

Since the size of the experiment is large, we exclusively present the results obtained considering the PM as fuzzy measure, due to the fact that PM obtained a superior performance in the previous analyses. The statistical results related with the other fuzzy measures are available online in the git repository, which follow the same structure of Table VII.

To facilitate the understanding, we provide, in Table VIII, a summary of the results obtained in the statistical analyses for all fuzzy measures. In this table, the rows are the different dC_F -integrals and the columns the considered fuzzy measures. The value of each cells is the number of times that the combination of restricted dissimilarity function and fuzzy measure was considered as control variable. For the Drastic Product and the Sin functions all cases are considered equivalent, thus, they are not considered. Finally, we present in the last row of the number of times ($\#nDiff$) that the dC_F -integrals have statistical difference (using different fuzzy measures) outperforming the baseline approach, δ_0 , which is Choquet-based and makes use of the difference operator.

Analyzing the number of times that different dC_F -integrals are used as control variable, its noticeable that there are always a restricted dissimilarity function that is superior than the

baseline approach. When the δ_5 is analyzed, this approach was selected as control variable in 11 different functions (Cardinality and Dirac). A large number of cases are also observable when the Wmean and PM are considered. On the other hand, in relation to the OWA, the number of functions selected are low. In fact, this is the only case where the number of dissimilarities used as control variable is not superior than the baseline. However, we have a draw between the dissimilarity function δ_2 and the baseline.

Taking a look in the obtained means and the statistical tests its easy to notice that the usage of dC_F -integrals are an interesting approach to use in alternative to the C_F -integrals. In a closer look, considering all the fuzzy measures, its noticeable that there are many approaches which are statistical different to the baseline. When the Wmean is used as fuzzy measure 16 out of the 21 considered functions presented statistical differences. For the PM there are 9 different functions. Only for OWA this behavior was not confirmed.

C. A deeper analysis on dC_F -integrals

The previous analysis have shown the dC_F -integrals performance and when it is an improvement over the C_F -integrals. In this subsection, we present the last part of the study, which aims to investigate which restricted dissimilarity functions best fit with which functions F and fuzzy measures.

In Table IX we summarize for each function (rows) and fuzzy measure (columns), the cases that are considered as control variable by the Aligned Friedman rank test computed in the last subsection.

Making an analysis of this table per function, it is noticeable a trend for the restricted dissimilarity functions. In fact, in almost all cases there was at least 3 or more (out of 5 fuzzy measures) restricted dissimilarity functions used. Only when the O_{RS} is considered, the baseline δ_0 approach is considered as control variable more than the dC_F -integrals. As mentioned before, the performance of the Drastic product and Sin functions are similar in all cases.

This affirmation also holds when an analysis is done by each fuzzy measure. In this case it is noticeable the dominance of the dissimilarity functions δ_5 and δ_2 in all cases. However it is also observable that the baseline approach is considered as control method in some cases, appearing as control variable more than the dissimilarity functions δ_1 , δ_3 and δ_4 .

VII. CONCLUSION

In this paper the concept of dC_F -integrals was introduced. Precisely, these functions generalize the C_F -integrals [14] by restricted dissimilarity functions δ [24], that is, we replaced the difference operator used by the C_F -integrals by restricted dissimilarity functions. Also, dC_F -integrals can be understood as a generalization of the d-Choquet integral [25] by a function F . We have shown some important properties that the dC_F -integrals satisfy, which are based on characteristics of the function F and the restricted dissimilarity functions.

The dC_F -integrals were applied as the aggregation-like operator in the FRM of FTBCs, in a very large experiment, with different analyses, considering several points of view.

TABLE VII: Friedman rank tests and APV considering PM as fuzzy measure.

Method	T_M Rank	APV	Method	T_P Rank	APV	Method	T_L Rank	APV	Method	T_{DP} Rank	APV	Method	T_{NM} Rank	APV	Method	T_{HP} Rank	APV	Method	O_B Rank	APV
δ_5	58.82	(-)	δ_2	57.79	(-)	δ_5	60.80	(-)	C_{TDP}	99.50	(-)	δ_2	58.23	(-)	δ_5	68.09	(-)	δ_2	62.14	(-)
δ_2	69.45	0.45	δ_5	72.91	0.28	δ_2	71.32	0.45	δ_1	99.50	1.00	δ_5	62.61	0.75	δ_2	69.32	0.93	δ_3	85.06	0.19
δ_3	94.20	0.02	δ_3	89.06	0.05	δ_3	104.00	0.00	δ_2	99.50	1.00	C_{TNM}	84.26	0.13	C_{THP}	83.39	0.55	δ_5	85.68	0.19
C_{TM}	100.53	0.00	C_{TP}	106.03	0.00	C_{TL}	105.38	0.00	δ_3	99.50	1.00	δ_3	113.92	0.00	δ_3	94.23	0.19	C_{OB}	96.02	0.04
δ_4	127.32	0.00	δ_4	132.89	0.00	δ_4	112.06	0.00	δ_5	99.50	1.00	δ_4	132.76	0.00	δ_4	131.38	0.00	δ_4	121.38	0.00
δ_1	146.68	0.00	δ_1	138.32	0.00	δ_1	143.44	0.00	δ_5	99.50	1.00	δ_4	145.23	0.00	δ_1	150.59	0.00	δ_1	146.73	0.00
Method	O_{mM} Rank	APV	Method	O_α Rank	APV	Method	O_{div} Rank	APV	Method	GM Rank	APV	Method	HM Rank	APV	Method	Sin Rank	APV	Method	O_{RS} Rank	APV
δ_5	61.98	(-)	δ_2	71.11	(-)	δ_5	69.14	(-)	δ_2	85.05	(-)	C_{HM}	72.26	(-)	C_{Sin}	99.50	(-)	C_{ORS}	81.62	(-)
δ_2	65.94	0.77	δ_5	72.21	1.00	δ_2	69.59	0.97	C_{GM}	93.08	1.00	δ_5	78.06	1.00	δ_1	99.50	1.00	δ_2	93.59	0.83
C_{OmM}	99.06	0.01	δ_3	79.26	1.00	δ_3	88.53	0.33	δ_5	95.26	1.00	δ_2	78.68	1.00	δ_2	99.50	1.00	δ_4	94.33	0.83
δ_3	107.14	0.00	C_{O_α}	103.15	0.06	$C_{O_{div}}$	100.18	0.08	δ_3	96.32	1.00	δ_3	92.89	0.43	δ_3	99.50	1.00	δ_5	96.88	0.83
δ_4	118.29	0.00	δ_4	122.73	0.00	δ_4	126.74	0.00	δ_4	101.61	0.96	δ_4	128.30	0.00	δ_4	99.50	1.00	δ_3	99.71	0.79
δ_1	144.59	0.00	δ_1	148.55	0.00	δ_1	142.82	0.00	δ_1	125.70	0.01	δ_1	146.80	0.00	δ_5	99.50	1.00	δ_1	130.86	0.00
Method	C_F Rank	APV	Method	C_L Rank	APV	Method	F_{GL} Rank	APV	Method	F_{BPC} Rank	APV	Method	F_{BD1} Rank	APV	Method	F_{NA} Rank	APV	Method	F_{NA2} Rank	APV
δ_5	68.06	(-)	δ_5	68.52	(-)	δ_2	80.82	(-)	δ_2	60.89	(-)	δ_5	59.65	(-)	δ_2	70.24	(-)	C_{FNA2}	73.39	(-)
δ_2	77.45	0.50	δ_2	68.89	0.97	δ_5	87.12	1.00	δ_5	68.58	0.58	C_{FBD1}	80.82	0.21	δ_5	79.74	0.51	δ_1	89.58	0.36
δ_3	88.85	0.28	δ_3	80.21	0.81	δ_4	90.05	1.00	C_{FBPC}	95.89	0.03	δ_2	82.45	0.21	C_{FNA}	86.21	0.51	δ_4	92.11	0.36
C_{CF}	94.38	0.18	C_{CL}	104.80	0.03	C_{FGL}	95.97	0.84	δ_3	96.06	0.03	δ_3	98.88	0.01	δ_3	94.23	0.26	δ_3	95.70	0.34
δ_4	126.85	0.00	δ_4	122.24	0.00	δ_3	107.86	0.22	δ_4	132.97	0.00	δ_4	125.18	0.00	δ_4	113.08	0.00	δ_2	120.73	0.00
δ_1	141.41	0.00	δ_1	152.33	0.00	δ_1	135.18	0.00	δ_1	142.61	0.00	δ_1	150.02	0.00	δ_1	153.50	0.00	δ_5	125.50	0.00

TABLE VIII: Total of times that each approach is considered as control variable in the Friedman rank test

	Cardinality	Dirac	OWA	Wmean	PM
δ_0	—	5	8	1	3
δ_1	—	—	—	1	—
δ_2	5	2	8	7	8
δ_3	2	1	—	1	—
δ_4	1	—	—	—	—
δ_5	11	11	3	9	8
#nDiff	8	5	0	16	9

TABLE IX: Summary of the generalizations that are considered as control method by the Friedman align rank test

	Card	Dirac	OWA	Wmean	PM
T_M	δ_5	δ_0	δ_5	δ_2	δ_5
T_P	δ_5	δ_5	δ_2	δ_2	δ_2
T_L	δ_5	δ_5	δ_0	δ_2	δ_5
T_{DP}	δ_0	δ_5	δ_0	δ_0	δ_0
T_{NM}	δ_5	δ_5	δ_0	δ_5	δ_2
T_{HP}	δ_5	δ_0	δ_2	δ_5	δ_5
O_B	δ_2	δ_5	δ_2	δ_5	δ_2
O_{mM}	δ_5	δ_5	δ_5	δ_5	δ_5
O_α	δ_5	δ_5	δ_2	δ_5	δ_2
O_{div}	δ_5	δ_5	δ_0	δ_5	δ_5
GM	δ_3	δ_0	δ_0	δ_3	δ_2
HM	δ_2	δ_0	δ_2	δ_2	δ_0
Sin	δ_0	δ_0	δ_0	δ_0	δ_0
O_{RS}	δ_3	δ_0	δ_0	δ_2	δ_0
C_F	δ_5	δ_5	δ_2	δ_2	δ_5
C_L	δ_2	δ_0	δ_0	δ_5	δ_5
F_{GL}	δ_5	δ_3	δ_0	δ_0	δ_2
F_{BPC}	δ_5	δ_5	δ_2	δ_5	δ_2
F_{BD1}	δ_2	δ_2	δ_5	δ_2	δ_5
F_{NA}	δ_2	δ_5	δ_2	δ_5	δ_2
F_{NA2}	δ_4	δ_2	δ_0	δ_1	δ_0

Having into account the obtained results, we can notice that the dC_F -integrals could be considered as a good alternative to be used instead of the C_F -integrals in classification problems. This is due to the fact that in several different cases the performance of such integrals are superior with statistical difference, having into account various fuzzy measures, when compared with the baseline approach.

In a more inclusive scenario, our developments showed that the dC_F -integrals can possibly enlarge the applicability of C_F -integrals, since we can use different combinations of restricted dissimilarity functions, functions F and fuzzy measures, which can be adapted to each kind of application and context.

Future works are concerned with the usage of the dC_F -integrals in the context of multi-criteria decision making. The study of other properties is also ongoing work.

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5 dC_F -integrals Applied to GMC-RTOPSIS

In this chapter we present a new separation measure based on the dC_F -integral introduced in last Chapter. Additionally, we apply it in the GMC-RTOPSIS [26] and use the same problem of choosing a supplier presented in the Chapters 2 and 3. The following results are intended to be submitted to the journal Information Fusion¹.

5.1 A new separation measure

From the definition of dC_F -integral proposed in Chapter 4 we introduce the dC_F -separation measure by:

Definition 5.1.1 (dC_F -separation measure). Let $F : [0, 1]^2 \rightarrow [0, 1]$ be a bivariate function, $\delta : [0, 1]^2 \rightarrow [0, 1]$ a restricted dissimilarity and $m : 2^N \rightarrow [0, 1]$ a fuzzy measure. Then, a dC_F -separation measure $S^* : [0, 1]^2 \rightarrow [0, 1]$ is defined, for all $i \in \{1, \dots, m\}$, by the equations:

$$\begin{aligned} S_i^+(\mathbf{Y}) &= \sqrt{\min \left\{ 1, \sum_{j=1}^n F \left(\delta \left(d_{i(j)}^+, d_{i(j-1)}^+ \right), m_Y \left(\mathbf{C}_{(j)}^+ \right) \right) \right\}} \\ S_i^-(\mathbf{Y}) &= \sqrt{\min \left\{ 1, \sum_{j=1}^n F \left(\delta \left(d_{i(j)}^-, d_{i(j-1)}^- \right), m_Y \left(\mathbf{C}_{(j)}^- \right) \right) \right\}} \end{aligned} \quad (1)$$

where $d_{i(1)}^+ \leq \dots \leq d_{i(n)}^+$, $d_{i(1)}^- \leq \dots \leq d_{i(n)}^-$, for each $j \in \{1, \dots, n-1\}$, $\mathbf{C}_{(j)}^+$ is the criterion correspondent to $d_{i(j)}^+$, $\mathbf{C}_{(j)}^-$ is the criterion correspondent to $d_{i(j)}^-$, $\mathbf{C}_{(j)}^+ = \{C_{(j)}^+, C_{(j+1)}^+, \dots, C_{(n)}^+\}$, $\mathbf{C}_{(j)}^- = \{C_{(j)}^-, C_{(j+1)}^-, \dots, C_{(n)}^-\}$, $\mathbf{C}_{(n+1)}^+ = \mathbf{C}_{(n+1)}^- = \emptyset$ and m_Y is the learned fuzzy measure by a Particle Swarm Optimization algorithm [36].

5.2 Methodology

We applied the dC_F -separation measure in the same environment of C_T - and CC-separation measures from Chapters 2 and 3. The problem definition and the steps of the

¹<https://www.sciencedirect.com/journal/information-fusion>

algorithm are detailed in Chapter 3 Sections 4.1 and 4.2. The parameters of the simulation were, also, 30 particles for the PSO and 100 interactions. We executed the method for each one of the 294 combinations of function F , from Table 21, and δ function, from Table 22.

To compare the results and select the best method for the problem, we use the definition of difference between the $\hat{c}_i$ values from the alternatives ranked first and second from Chapter 3, which is:

$$\Delta_{R1,R2} = \max(\hat{c}_1) - \max(\hat{c}_2) \quad (2)$$

where $\hat{c}_1 = \{\hat{c}_i \mid i \in \{1, \dots, m\}\}$ and $\hat{c}_2 = \hat{c}_1 - \{\max(\hat{c}_1)\}$.

5.3 Results

Table 23, at the end of this Chapter, presents the results of the all combinations of functions F and δ from Tables 21 and 22, ordered by highest differences $\Delta_{R1,R2}$. We will use the notation (F, δ) to explicit the combination of functions F and dissimilarities.

Notice that the results from the C_T -integrals in Chapter 2 were using the dissimilarity function δ_4 . Therefore, they are in the form of (T, δ_4) .

One can see the top 6 lines of the Table 23 are from the T_{DP} t-norm, with all the tested dissimilarity functions. This is from the radical definition of the function. It only attributed non-zero values for its best possible solution, while zeroing all the others alternatives. Although they resulted in the biggest differences between the alternatives ranked first and second, we will not consider them, seeing that it totally ignores the other places of the rank. Despite this, we kept them for verification proposes.

On the other side of the table, the last nineteen differences resulted in zero, i.e. $\Delta_{R1,R2} = 0$. For those combinations of functions F and δ , for the proposed problem, the PSO algorithm seems to be unable to calculate the fuzzy measure correctly. This results in a the dC_F -separation measure being 0.5 for all the alternatives (with a minimal divergence in some alternatives).

After disconsidering the T_{DP} results because of the above reasons, we have the Łukasiewicz t-norm combined with the squared dissimilarity δ_1 obtained the highest difference with $\Delta_{R1,R2} = 0.1352$. Also, the Łukasiewicz t-norm have achieved the highest separation between the alternatives ranked first and second when using C_T - and CC -separation measures. Close to the Łukasiewicz t-norm we have most of the F_{BD1} function, with various q parameters and different functions δ . The difference among the $\Delta_{R1,R2}$'s of the F_{BD1} from the top of the table ranged from 0.1341 to 0.0526. Note how the function F_{BD1} seems to assign a higher weight for the alternative 4 (A_4) in detriment of alternative 3 (A_3), which resulted in a inversion of the top ranking, in relation to the (T_L, δ_1) combination, with $A_4 \geq A_3$ and sometimes even $A_4 \geq A_1$.

Table 21: Definition of the F functions used in this Master's thesis.

Function definition	Description
$T_M(x, y) = \min\{x, y\}$	Minimum t-norm
$T_P(x, y) = xy$	Algebraic product
$T_L(x, y) = \max\{0, x + y - 1\}$	Łukasiewicz
$T_{DP}(x, y) = \begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$	Drastic Product
$T_{NM}(x, y) = \begin{cases} \min\{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$	Nilpotent Minimum
$T_{HP}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$	Hamacher Product
$O_B(x, y) = \min\{x\sqrt{y}, y\sqrt{x}\}$	[8], Cuadras-Augé copula [31]
$O_{mM}(x, y) = \min\{x, y\} \max\{x^2, y^2\}$	[13, 17, 16]
$O_\alpha(x, y) = xy(1 + \alpha(1 - x)(1 - y))$, with $\alpha \in [-1, 1] \setminus \{0\}$	[1], Farlie-Gumbel-Morgenstern copula family
$O_{Div}(x, y) = \frac{xy + \min\{x, y\}}{2}$	[1, 28]
$GM(x, y) = \sqrt{xy}$	Geometric Mean, [19]
$HM(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \\ \frac{2}{\frac{1}{x} + \frac{1}{y}} & \text{otherwise.} \end{cases}$	Harmonic Mean, [19]
$Sin(x, y) = \sin\left(\frac{\pi}{2}(xy)^{\frac{1}{4}}\right)$	Sine, [19]
$O_{RS}(x, y) = \min\left\{\frac{(x+1)\sqrt{y}}{2}, y\sqrt{x}\right\}$	
$C_F(x, y) = xy + x^2y(1 - x)(1 - y)$	[25, 28]
$C_L(x, y) = \max\{\min\{x, \frac{y}{2}\}, x + y - 1\}$	[1, 28]
$F_{GL}(x, y) = \sqrt{\frac{x(y+1)}{2}}$	
$F_{BPC}(x, y) = xy^2$	[2]
$F_{BD1}(x, y) = \min\{x, 1 - x + \min\{x, y^q\}\}$, with $0 < q \leq 1$	[29, 30]
$F_{NA}(x, y) = \begin{cases} x & \text{if } x \leq y \\ \min\{\frac{x}{2}, y\} & \text{otherwise.} \end{cases}$	[29, 30]
$F_{NA2}(x, y) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{x+y}{2} & \text{if } 0 < x \leq y \\ \min\{\frac{x}{2}, y\} & \text{otherwise.} \end{cases}$	[29, 30]

In the middle of the F_{BD1} are other functions combinations, like the (T_{NM}, δ_4) with $\Delta_{R1,R2} = 0.1177$, (T_L, δ_4) with $\Delta_{R1,R2} = 0.0949$ and (T_{NM}, δ_1) with $\Delta_{R1,R2} = 0.0942$.

The O_α functions family, which achieved one of the highest difference when used in the CC -separation measure, are spread through the table. Although, the highest dif-

Table 22: Dissimilarity functions δ used in this dissertation.

Dissimilarities
$\delta_0(x, y) = x - y $
$\delta_1(x, y) = (x - y)^2$
$\delta_2(x, y) = \sqrt{ x - y }$
$\delta_3(x, y) = \sqrt{x} - \sqrt{y} $
$\delta_4(x, y) = x^2 - y^2 $
$\delta_5(x, y) = (\sqrt{x} - \sqrt{y})^2$

ferences achieved are quite distant from the best one, with only $\Delta_{R1,R2} = 0.0412$ and $\Delta_{R1,R2} = 0.0411$ when combined with δ_4 and alpha parameters as $\alpha = -0.5$ and $\alpha = 0.2$, respectively. Also, even most of them ranking $A_3 \geq A_4$, some have inverted, and ranked $A_4 \geq A_3$, like the $(O_{\alpha=0.9}, \delta_1)$ and $(O_{\alpha=0.9}, \delta_5)$.

5.4 Discussion

In this master's thesis, for the GMC-RTOPSIS, we used the difference between the values of the alternatives ranked first and second, $\Delta_{R1,R2}$. Although it does not give the “full picture” of the rank, it is a goal-oriented measure. It is able to given a objective answer on which combination (F, δ) to choose when running multiple of them.

Even though we do not analyze, there are other methods, like using the rank which most appear in the results. For example: in our ranking there were the following quantity of ranked first and second alternatives: $A_3 \geq A_1$ were 7; $A_3 \geq A_4$ were 121; $A_4 \geq A_1$ were 10; and $A_4 \geq A_3$ there were 137 combinations of (F, δ) . Therefore, if taking into account this method, we would choose the alternative 4 instead of the alternative 3. Also, one may consider a combination of both, using the difference $\Delta_{R1,R2}$ based on a weight given by the commonness of the rank.

The choice of the method may also depend on the type of problem/data we have. But it is not the intention of this thesis to answer this statement, and will be analyzed in future works.

Table 23: Rankings and differences $\Delta_{R1,R2}$ of the problem by using dC_F -integral with several combinations for F and δ functions.

F	δ	Ranked 1st	Ranked 2nd	Ranked 3rd	Ranked 4th	$\Delta_{R1,R2}$
T_{DP}	δ_0	$A_3(0.5573)$	$A_1(0.0000)$	$A_2(0.0000)$	$A_4(0.0000)$	0.5573
T_{DP}	δ_1	$A_3(0.5202)$	$A_1(0.0000)$	$A_2(0.0000)$	$A_4(0.0000)$	0.5202
T_{DP}	δ_4	$A_3(0.5093)$	$A_1(0.0000)$	$A_2(0.0000)$	$A_4(0.0000)$	0.5093
T_{DP}	δ_2	$A_3(0.4453)$	$A_1(0.0000)$	$A_2(0.0000)$	$A_4(0.0000)$	0.4453
T_{DP}	δ_3	$A_3(0.4146)$	$A_1(0.0000)$	$A_2(0.0000)$	$A_4(0.0000)$	0.4146
T_{DP}	δ_5	$A_3(0.3668)$	$A_1(0.0000)$	$A_2(0.0000)$	$A_4(0.0000)$	0.3668
T_L	δ_1	$A_3(0.6593)$	$A_1(0.5240)$	$A_4(0.5172)$	$A_2(0.4848)$	0.1352
$F_{BD1,q=0.2}$	δ_4	$A_4(0.6769)$	$A_3(0.5429)$	$A_1(0.3844)$	$A_2(0.2688)$	0.1341
$F_{BD1,q=0.1}$	δ_4	$A_4(0.6761)$	$A_3(0.5422)$	$A_1(0.3852)$	$A_2(0.2698)$	0.1339
$F_{BD1,q=0.3}$	δ_4	$A_4(0.6751)$	$A_3(0.5436)$	$A_1(0.3831)$	$A_2(0.2699)$	0.1314
$F_{BD1,q=0.4}$	δ_4	$A_4(0.6678)$	$A_3(0.5443)$	$A_1(0.3817)$	$A_2(0.2753)$	0.1235
T_{NM}	δ_4	$A_3(0.6696)$	$A_4(0.5519)$	$A_1(0.5207)$	$A_2(0.4173)$	0.1177
$F_{BD1,q=0.5}$	δ_4	$A_4(0.6622)$	$A_3(0.5450)$	$A_1(0.3804)$	$A_2(0.2796)$	0.1172
$F_{BD1,q=0.1}$	δ_1	$A_4(0.6402)$	$A_3(0.5245)$	$A_1(0.4350)$	$A_2(0.3425)$	0.1157
$F_{BD1,q=0.2}$	δ_1	$A_4(0.6402)$	$A_3(0.5245)$	$A_1(0.4350)$	$A_2(0.3425)$	0.1157
$F_{BD1,q=0.3}$	δ_1	$A_4(0.6402)$	$A_3(0.5245)$	$A_1(0.4350)$	$A_2(0.3425)$	0.1157
$F_{BD1,q=0.4}$	δ_1	$A_4(0.6402)$	$A_3(0.5245)$	$A_1(0.4350)$	$A_2(0.3425)$	0.1157
$F_{BD1,q=0.5}$	δ_1	$A_4(0.6402)$	$A_3(0.5245)$	$A_1(0.4350)$	$A_2(0.3425)$	0.1157
$F_{BD1,q=0.6}$	δ_1	$A_4(0.6402)$	$A_3(0.5245)$	$A_1(0.4350)$	$A_2(0.3425)$	0.1157
$F_{BD1,q=0.7}$	δ_1	$A_4(0.6402)$	$A_3(0.5245)$	$A_1(0.4350)$	$A_2(0.3425)$	0.1157
$F_{BD1,q=0.8}$	δ_1	$A_4(0.6402)$	$A_3(0.5246)$	$A_1(0.4350)$	$A_2(0.3425)$	0.1156
$F_{BD1,q=0.9}$	δ_1	$A_4(0.6402)$	$A_3(0.5246)$	$A_1(0.4350)$	$A_2(0.3425)$	0.1156
$F_{BD1,q=1}$	δ_1	$A_4(0.6402)$	$A_3(0.5247)$	$A_1(0.4350)$	$A_2(0.3425)$	0.1155
$F_{BD1,q=0.6}$	δ_4	$A_4(0.6581)$	$A_3(0.5457)$	$A_1(0.3795)$	$A_2(0.2828)$	0.1124
$F_{BD1,q=0.7}$	δ_4	$A_4(0.6551)$	$A_3(0.5462)$	$A_1(0.3788)$	$A_2(0.2852)$	0.1089
$F_{BD1,q=0.8}$	δ_4	$A_4(0.6531)$	$A_3(0.5466)$	$A_1(0.3782)$	$A_2(0.2868)$	0.1065
$F_{BD1,q=0.9}$	δ_4	$A_4(0.6517)$	$A_3(0.5469)$	$A_1(0.3779)$	$A_2(0.2880)$	0.1048
$F_{BD1,q=1}$	δ_4	$A_4(0.6507)$	$A_3(0.5472)$	$A_1(0.3776)$	$A_2(0.2888)$	0.1035
T_L	δ_4	$A_3(0.6535)$	$A_4(0.5586)$	$A_1(0.4528)$	$A_2(0.4110)$	0.0949
T_{NM}	δ_1	$A_3(0.6168)$	$A_4(0.5226)$	$A_2(0.4860)$	$A_1(0.4525)$	0.0942
$F_{BD1,q=0.1}$	δ_0	$A_4(0.5928)$	$A_3(0.5231)$	$A_1(0.4403)$	$A_2(0.3759)$	0.0697
$F_{BD1,q=0.2}$	δ_0	$A_4(0.5928)$	$A_3(0.5231)$	$A_1(0.4403)$	$A_2(0.3759)$	0.0697
$F_{BD1,q=0.3}$	δ_0	$A_4(0.5929)$	$A_3(0.5238)$	$A_1(0.4401)$	$A_2(0.3757)$	0.0691
$F_{BD1,q=0.4}$	δ_0	$A_4(0.5931)$	$A_3(0.5249)$	$A_1(0.4398)$	$A_2(0.3753)$	0.0682

Table 23: Rankings and differences $\Delta_{R1,R2}$ of the problem by using dC_F -integral with several combinations for F and δ functions.

F	δ	Ranked 1st	Ranked 2nd	Ranked 3rd	Ranked 4th	$\Delta_{R1,R2}$
$F_{BD1,q=0.5}$	δ_0	$A_4(0.5935)$	$A_3(0.5257)$	$A_1(0.4395)$	$A_2(0.3748)$	0.0677
$F_{BD1,q=0.6}$	δ_0	$A_4(0.5938)$	$A_3(0.5263)$	$A_1(0.4393)$	$A_2(0.3744)$	0.0675
$F_{BD1,q=0.7}$	δ_0	$A_4(0.5941)$	$A_3(0.5267)$	$A_1(0.4390)$	$A_2(0.3740)$	0.0673
$F_{BD1,q=0.8}$	δ_0	$A_4(0.5943)$	$A_3(0.5270)$	$A_1(0.4388)$	$A_2(0.3738)$	0.0673
$F_{BD1,q=0.9}$	δ_0	$A_4(0.5944)$	$A_3(0.5272)$	$A_1(0.4386)$	$A_2(0.3736)$	0.0672
$F_{BD1,q=1}$	δ_0	$A_4(0.5946)$	$A_3(0.5274)$	$A_1(0.4385)$	$A_2(0.3734)$	0.0671
$F_{BD1,q=0.1}$	δ_5	$A_4(0.5531)$	$A_3(0.5005)$	$A_1(0.4829)$	$A_2(0.4355)$	0.0526
$F_{BD1,q=0.2}$	δ_5	$A_4(0.5531)$	$A_3(0.5005)$	$A_1(0.4829)$	$A_2(0.4355)$	0.0526
$F_{BD1,q=0.3}$	δ_5	$A_4(0.5531)$	$A_3(0.5005)$	$A_1(0.4829)$	$A_2(0.4355)$	0.0526
$F_{BD1,q=0.4}$	δ_5	$A_4(0.5531)$	$A_3(0.5005)$	$A_1(0.4829)$	$A_2(0.4355)$	0.0526
$F_{BD1,q=0.5}$	δ_5	$A_4(0.5531)$	$A_3(0.5005)$	$A_1(0.4829)$	$A_2(0.4355)$	0.0526
$F_{BD1,q=0.6}$	δ_5	$A_4(0.5531)$	$A_3(0.5005)$	$A_1(0.4829)$	$A_2(0.4355)$	0.0526
$F_{BD1,q=0.7}$	δ_5	$A_4(0.5531)$	$A_3(0.5005)$	$A_1(0.4829)$	$A_2(0.4355)$	0.0526
$F_{BD1,q=0.8}$	δ_5	$A_4(0.5531)$	$A_3(0.5005)$	$A_1(0.4829)$	$A_2(0.4355)$	0.0526
$F_{BD1,q=0.9}$	δ_5	$A_4(0.5531)$	$A_3(0.5005)$	$A_1(0.4829)$	$A_2(0.4355)$	0.0526
$F_{BD1,q=1}$	δ_5	$A_4(0.5531)$	$A_3(0.5005)$	$A_1(0.4829)$	$A_2(0.4355)$	0.0526
T_M	δ_4	$A_4(0.5444)$	$A_3(0.4989)$	$A_1(0.4428)$	$A_2(0.3864)$	0.0454
$O_{\alpha=-0.5}$	δ_4	$A_3(0.6018)$	$A_4(0.5606)$	$A_1(0.4444)$	$A_2(0.4104)$	0.0412
$O_{\alpha=0.2}$	δ_4	$A_3(0.5990)$	$A_4(0.5579)$	$A_1(0.4507)$	$A_2(0.4119)$	0.0411
F_{GL}	δ_1	$A_4(0.5777)$	$A_3(0.5383)$	$A_1(0.4574)$	$A_2(0.3930)$	0.0394
$O_{\alpha=0.1}$	δ_4	$A_3(0.5927)$	$A_4(0.5539)$	$A_1(0.4652)$	$A_2(0.4035)$	0.0387
$O_{\alpha=0.6}$	δ_4	$A_3(0.5948)$	$A_4(0.5565)$	$A_1(0.4536)$	$A_2(0.4064)$	0.0383
T_{NM}	δ_0	$A_3(0.5716)$	$A_4(0.5338)$	$A_1(0.4605)$	$A_2(0.4584)$	0.0377
F_{NA}	δ_5	$A_4(0.5434)$	$A_3(0.5065)$	$A_1(0.4731)$	$A_2(0.4435)$	0.0369
$O_{\alpha=-0.9}$	δ_4	$A_3(0.6064)$	$A_4(0.5696)$	$A_1(0.4479)$	$A_2(0.3932)$	0.0368
$O_{\alpha=-1}$	δ_4	$A_3(0.5977)$	$A_4(0.5621)$	$A_1(0.4343)$	$A_2(0.4029)$	0.0356
$O_{\alpha=0.7}$	δ_4	$A_3(0.5941)$	$A_4(0.5584)$	$A_1(0.4684)$	$A_2(0.4011)$	0.0356
$F_{BD1,q=0.1}$	δ_3	$A_4(0.5471)$	$A_3(0.5119)$	$A_1(0.4698)$	$A_2(0.4366)$	0.0352
$F_{BD1,q=0.2}$	δ_3	$A_4(0.5471)$	$A_3(0.5119)$	$A_1(0.4698)$	$A_2(0.4366)$	0.0352
$F_{BD1,q=0.3}$	δ_3	$A_4(0.5471)$	$A_3(0.5119)$	$A_1(0.4698)$	$A_2(0.4366)$	0.0352
$F_{BD1,q=0.4}$	δ_3	$A_4(0.5471)$	$A_3(0.5119)$	$A_1(0.4698)$	$A_2(0.4366)$	0.0352
$F_{BD1,q=0.5}$	δ_3	$A_4(0.5471)$	$A_3(0.5119)$	$A_1(0.4698)$	$A_2(0.4366)$	0.0352
$F_{BD1,q=0.6}$	δ_3	$A_4(0.5471)$	$A_3(0.5119)$	$A_1(0.4698)$	$A_2(0.4366)$	0.0352
$F_{BD1,q=0.7}$	δ_3	$A_4(0.5471)$	$A_3(0.5119)$	$A_1(0.4698)$	$A_2(0.4366)$	0.0352
$F_{BD1,q=0.8}$	δ_3	$A_4(0.5471)$	$A_3(0.5119)$	$A_1(0.4698)$	$A_2(0.4366)$	0.0352

Table 23: Rankings and differences $\Delta_{R1,R2}$ of the problem by using dC_F -integral with several combinations for F and δ functions.

F	δ	Ranked 1st	Ranked 2nd	Ranked 3rd	Ranked 4th	$\Delta_{R1,R2}$
$F_{BD1,q=0.9}$	δ_3	$A_4(0.5471)$	$A_3(0.5119)$	$A_1(0.4698)$	$A_2(0.4366)$	0.0352
$F_{BD1,q=1}$	δ_3	$A_4(0.5471)$	$A_3(0.5119)$	$A_1(0.4698)$	$A_2(0.4366)$	0.0352
T_{HP}	δ_4	$A_4(0.5593)$	$A_3(0.5245)$	$A_1(0.4279)$	$A_2(0.3868)$	0.0348
$O_{\alpha=-0.8}$	δ_4	$A_3(0.6071)$	$A_4(0.5731)$	$A_1(0.4232)$	$A_2(0.4074)$	0.0340
GM	δ_4	$A_4(0.5642)$	$A_3(0.5303)$	$A_1(0.4418)$	$A_2(0.4016)$	0.0339
C_F	δ_4	$A_3(0.5944)$	$A_4(0.5608)$	$A_1(0.4436)$	$A_2(0.4053)$	0.0336
$O_{\alpha=0.8}$	δ_4	$A_3(0.5822)$	$A_4(0.5490)$	$A_1(0.4698)$	$A_2(0.4039)$	0.0332
F_{NA}	δ_0	$A_4(0.5429)$	$A_3(0.5100)$	$A_1(0.4565)$	$A_2(0.4211)$	0.0329
$O_{\alpha=1}$	δ_4	$A_3(0.5893)$	$A_4(0.5569)$	$A_1(0.4702)$	$A_2(0.4003)$	0.0324
O_{mM}	δ_4	$A_3(0.6002)$	$A_4(0.5681)$	$A_1(0.4482)$	$A_2(0.4019)$	0.0321
T_M	δ_5	$A_4(0.5436)$	$A_3(0.5118)$	$A_1(0.4726)$	$A_2(0.4429)$	0.0318
F_{NA}	δ_1	$A_4(0.5575)$	$A_3(0.5261)$	$A_1(0.4401)$	$A_2(0.4210)$	0.0315
$O_{\alpha=-0.7}$	δ_4	$A_3(0.6033)$	$A_4(0.5726)$	$A_1(0.4313)$	$A_2(0.4027)$	0.0306
F_{BPC}	δ_4	$A_3(0.5924)$	$A_4(0.5635)$	$A_1(0.4501)$	$A_2(0.3999)$	0.0288
$O_{\alpha=0.4}$	δ_4	$A_3(0.5917)$	$A_4(0.5635)$	$A_1(0.4452)$	$A_2(0.4085)$	0.0282
T_L	δ_0	$A_3(0.5631)$	$A_4(0.5350)$	$A_1(0.4575)$	$A_2(0.4506)$	0.0282
$O_{\alpha=-0.4}$	δ_4	$A_3(0.5972)$	$A_4(0.5702)$	$A_1(0.4350)$	$A_2(0.3969)$	0.0270
$O_{\alpha=-0.6}$	δ_4	$A_3(0.5988)$	$A_4(0.5717)$	$A_1(0.4291)$	$A_2(0.3952)$	0.0270
T_M	δ_3	$A_4(0.5239)$	$A_3(0.4979)$	$A_1(0.4710)$	$A_2(0.4587)$	0.0260
C_L	δ_5	$A_4(0.5322)$	$A_3(0.5065)$	$A_1(0.4813)$	$A_2(0.4500)$	0.0257
$O_{\alpha=-0.3}$	δ_4	$A_3(0.5924)$	$A_4(0.5673)$	$A_1(0.4414)$	$A_2(0.4002)$	0.0252
$O_{\alpha=0.5}$	δ_4	$A_3(0.5823)$	$A_4(0.5573)$	$A_1(0.4488)$	$A_2(0.3987)$	0.0250
C_L	δ_0	$A_4(0.5320)$	$A_3(0.5074)$	$A_1(0.4592)$	$A_2(0.4382)$	0.0245
$O_{\alpha=0.9}$	δ_4	$A_3(0.5841)$	$A_4(0.5601)$	$A_1(0.4459)$	$A_2(0.4020)$	0.0241
C_L	δ_1	$A_4(0.5348)$	$A_3(0.5119)$	$A_1(0.4521)$	$A_2(0.4459)$	0.0229
O_B	δ_4	$A_3(0.5757)$	$A_4(0.5529)$	$A_1(0.4660)$	$A_2(0.4038)$	0.0228
HM	δ_4	$A_4(0.5568)$	$A_3(0.5341)$	$A_1(0.4323)$	$A_2(0.3885)$	0.0227
T_{NM}	δ_3	$A_3(0.5365)$	$A_4(0.5138)$	$A_2(0.4816)$	$A_1(0.4804)$	0.0227
$O_{\alpha=-0.2}$	δ_4	$A_3(0.5916)$	$A_4(0.5693)$	$A_1(0.4349)$	$A_2(0.4001)$	0.0223
$O_{\alpha=-1}$	δ_0	$A_3(0.5569)$	$A_4(0.5352)$	$A_1(0.4696)$	$A_2(0.4491)$	0.0217
F_{NA}	δ_4	$A_4(0.5415)$	$A_3(0.5200)$	$A_1(0.4514)$	$A_2(0.3991)$	0.0216
T_M	δ_1	$A_4(0.5622)$	$A_3(0.5417)$	$A_1(0.4339)$	$A_2(0.4183)$	0.0205
F_{GL}	δ_5	$A_4(0.5383)$	$A_3(0.5181)$	$A_1(0.4750)$	$A_2(0.4469)$	0.0202
O_B	δ_0	$A_3(0.5446)$	$A_4(0.5247)$	$A_1(0.4746)$	$A_2(0.4474)$	0.0199
HM	δ_1	$A_4(0.5512)$	$A_3(0.5315)$	$A_1(0.4508)$	$A_2(0.4328)$	0.0197

Table 23: Rankings and differences $\Delta_{R1,R2}$ of the problem by using dC_F -integral with several combinations for F and δ functions.

F	δ	Ranked 1st	Ranked 2nd	Ranked 3rd	Ranked 4th	$\Delta_{R1,R2}$
$O_{\alpha=0.3}$	δ_4	$A_3(0.5813)$	$A_4(0.5619)$	$A_1(0.4453)$	$A_2(0.4012)$	0.0194
T_{HP}	δ_1	$A_4(0.5507)$	$A_3(0.5316)$	$A_1(0.4548)$	$A_2(0.4341)$	0.0191
HM	δ_5	$A_4(0.5316)$	$A_3(0.5132)$	$A_1(0.4823)$	$A_2(0.4551)$	0.0184
$O_{\alpha=-0.3}$	δ_0	$A_3(0.5529)$	$A_4(0.5347)$	$A_1(0.4625)$	$A_2(0.4446)$	0.0182
T_{HP}	δ_5	$A_4(0.5333)$	$A_3(0.5152)$	$A_1(0.4856)$	$A_2(0.4492)$	0.0181
$O_{\alpha=-0.1}$	δ_4	$A_3(0.5888)$	$A_4(0.5719)$	$A_1(0.4330)$	$A_2(0.4001)$	0.0169
$O_{\alpha=-0.7}$	δ_0	$A_3(0.5544)$	$A_4(0.5378)$	$A_1(0.4622)$	$A_2(0.4475)$	0.0166
T_L	δ_3	$A_3(0.5333)$	$A_4(0.5170)$	$A_1(0.4797)$	$A_2(0.4770)$	0.0163
$O_{\alpha=-0.6}$	δ_0	$A_3(0.5525)$	$A_4(0.5377)$	$A_1(0.4585)$	$A_2(0.4475)$	0.0148
O_B	δ_1	$A_4(0.5491)$	$A_3(0.5345)$	$A_1(0.4556)$	$A_2(0.4513)$	0.0146
$O_{\alpha=0.8}$	δ_0	$A_3(0.5490)$	$A_4(0.5345)$	$A_1(0.4669)$	$A_2(0.4481)$	0.0145
O_{mM}	δ_0	$A_3(0.5534)$	$A_4(0.5391)$	$A_1(0.4630)$	$A_2(0.4430)$	0.0144
O_{Div}	δ_5	$A_4(0.5335)$	$A_3(0.5197)$	$A_1(0.4859)$	$A_2(0.4516)$	0.0139
$O_{\alpha=-0.8}$	δ_0	$A_3(0.5493)$	$A_4(0.5355)$	$A_1(0.4642)$	$A_2(0.4467)$	0.0138
F_{BPC}	δ_1	$A_3(0.5551)$	$A_4(0.5414)$	$A_2(0.4665)$	$A_1(0.4499)$	0.0137
$O_{\alpha=-0.1}$	δ_0	$A_3(0.5511)$	$A_4(0.5374)$	$A_1(0.4669)$	$A_2(0.4464)$	0.0137
O_{RS}	δ_1	$A_3(0.5510)$	$A_4(0.5373)$	$A_1(0.4614)$	$A_2(0.4475)$	0.0137
T_P	δ_0	$A_3(0.5510)$	$A_4(0.5373)$	$A_1(0.4614)$	$A_2(0.4475)$	0.0137
F_{GL}	δ_4	$A_4(0.5414)$	$A_3(0.5279)$	$A_1(0.4608)$	$A_2(0.4169)$	0.0135
T_M	δ_0	$A_4(0.5291)$	$A_3(0.5161)$	$A_1(0.4684)$	$A_2(0.4392)$	0.0131
$O_{\alpha=0.1}$	δ_0	$A_3(0.5501)$	$A_4(0.5371)$	$A_1(0.4659)$	$A_2(0.4459)$	0.0130
GM	δ_1	$A_3(0.5528)$	$A_4(0.5400)$	$A_1(0.4756)$	$A_2(0.4357)$	0.0128
F_{BPC}	δ_0	$A_3(0.5558)$	$A_4(0.5431)$	$A_1(0.4611)$	$A_2(0.4461)$	0.0127
$O_{\alpha=-0.5}$	δ_0	$A_3(0.5507)$	$A_4(0.5381)$	$A_1(0.4569)$	$A_2(0.4475)$	0.0126
T_L	δ_5	$A_3(0.5333)$	$A_4(0.5207)$	$A_1(0.5058)$	$A_2(0.4622)$	0.0126
$O_{\alpha=0.7}$	δ_0	$A_3(0.5459)$	$A_4(0.5334)$	$A_1(0.4676)$	$A_2(0.4473)$	0.0125
C_F	δ_0	$A_3(0.5501)$	$A_4(0.5380)$	$A_1(0.4583)$	$A_2(0.4456)$	0.0121
$O_{\alpha=0.2}$	δ_0	$A_3(0.5490)$	$A_4(0.5369)$	$A_1(0.4606)$	$A_2(0.4411)$	0.0121
T_P	δ_4	$A_3(0.5821)$	$A_4(0.5701)$	$A_1(0.4346)$	$A_2(0.3977)$	0.0120
$O_{\alpha=-0.7}$	δ_1	$A_3(0.5567)$	$A_4(0.5448)$	$A_2(0.4627)$	$A_1(0.4521)$	0.0119
$O_{\alpha=1}$	δ_0	$A_3(0.5481)$	$A_4(0.5361)$	$A_1(0.4665)$	$A_2(0.4500)$	0.0119
HM	δ_0	$A_4(0.5363)$	$A_3(0.5244)$	$A_1(0.4583)$	$A_2(0.4344)$	0.0119
T_{HP}	δ_0	$A_4(0.5336)$	$A_3(0.5222)$	$A_1(0.4645)$	$A_2(0.4371)$	0.0115
$O_{\alpha=-0.4}$	δ_0	$A_3(0.5501)$	$A_4(0.5388)$	$A_1(0.4592)$	$A_2(0.4462)$	0.0113
T_{NM}	δ_5	$A_3(0.5316)$	$A_4(0.5202)$	$A_1(0.5028)$	$A_2(0.4636)$	0.0113

Table 23: Rankings and differences $\Delta_{R1,R2}$ of the problem by using dC_F -integral with several combinations for F and δ functions.

F	δ	Ranked 1st	Ranked 2nd	Ranked 3rd	Ranked 4th	$\Delta_{R1,R2}$
$O_{\alpha=0.3}$	δ_0	$A_3(0.5459)$	$A_4(0.5352)$	$A_1(0.4611)$	$A_2(0.4436)$	0.0108
$O_{\alpha=-0.2}$	δ_0	$A_3(0.5482)$	$A_4(0.5375)$	$A_1(0.4587)$	$A_2(0.4442)$	0.0107
$O_{\alpha=0.1}$	δ_1	$A_4(0.5483)$	$A_3(0.5377)$	$A_2(0.4577)$	$A_1(0.4425)$	0.0106
$O_{\alpha=0.9}$	δ_1	$A_4(0.5439)$	$A_3(0.5334)$	$A_2(0.4591)$	$A_1(0.4512)$	0.0104
$O_{\alpha=0.9}$	δ_0	$A_3(0.5480)$	$A_4(0.5386)$	$A_1(0.4621)$	$A_2(0.4487)$	0.0094
O_B	δ_5	$A_4(0.5289)$	$A_3(0.5196)$	$A_1(0.4878)$	$A_2(0.4568)$	0.0093
$O_{\alpha=0.6}$	δ_0	$A_3(0.5467)$	$A_4(0.5375)$	$A_1(0.4629)$	$A_2(0.4453)$	0.0092
$O_{\alpha=-0.3}$	δ_3	$A_3(0.5269)$	$A_4(0.5181)$	$A_1(0.4796)$	$A_2(0.4733)$	0.0088
O_{Div}	δ_4	$A_3(0.5398)$	$A_4(0.5310)$	$A_1(0.4733)$	$A_2(0.4063)$	0.0088
O_{Div}	δ_3	$A_4(0.5212)$	$A_3(0.5127)$	$A_1(0.4750)$	$A_2(0.4663)$	0.0085
$O_{\alpha=0.4}$	δ_1	$A_4(0.5454)$	$A_3(0.5372)$	$A_2(0.4597)$	$A_1(0.4454)$	0.0082
T_{HP}	δ_3	$A_3(0.5269)$	$A_4(0.5187)$	$A_1(0.4863)$	$A_2(0.4692)$	0.0082
F_{BPC}	δ_3	$A_3(0.5276)$	$A_4(0.5196)$	$A_1(0.4821)$	$A_2(0.4721)$	0.0080
$O_{\alpha=-0.7}$	δ_3	$A_3(0.5261)$	$A_4(0.5181)$	$A_1(0.4789)$	$A_2(0.4730)$	0.0079
O_{Div}	δ_1	$A_3(0.5539)$	$A_4(0.5460)$	$A_1(0.4636)$	$A_2(0.4445)$	0.0079
$O_{\alpha=0.4}$	δ_0	$A_3(0.5452)$	$A_4(0.5374)$	$A_1(0.4633)$	$A_2(0.4457)$	0.0078
$O_{\alpha=-0.1}$	δ_1	$A_4(0.5486)$	$A_3(0.5409)$	$A_2(0.4599)$	$A_1(0.4428)$	0.0077
$O_{\alpha=-0.8}$	δ_1	$A_3(0.5552)$	$A_4(0.5478)$	$A_2(0.4632)$	$A_1(0.4480)$	0.0074
$O_{\alpha=-0.1}$	δ_3	$A_3(0.5263)$	$A_4(0.5192)$	$A_1(0.4796)$	$A_2(0.4713)$	0.0070
HM	δ_3	$A_4(0.5047)$	$A_3(0.4977)$	$A_1(0.4970)$	$A_2(0.4864)$	0.0070
$O_{\alpha=1}$	δ_5	$A_4(0.5267)$	$A_3(0.5198)$	$A_1(0.4930)$	$A_2(0.4592)$	0.0069
$O_{\alpha=-0.2}$	δ_1	$A_4(0.5490)$	$A_3(0.5422)$	$A_2(0.4599)$	$A_1(0.4395)$	0.0068
$O_{\alpha=-0.9}$	δ_0	$A_3(0.5472)$	$A_4(0.5406)$	$A_1(0.4587)$	$A_2(0.4435)$	0.0066
$O_{\alpha=-0.9}$	δ_3	$A_3(0.5262)$	$A_4(0.5196)$	$A_1(0.4828)$	$A_2(0.4736)$	0.0066
$O_{\alpha=-0.9}$	δ_1	$A_3(0.5522)$	$A_4(0.5456)$	$A_2(0.4602)$	$A_1(0.4457)$	0.0066
$O_{\alpha=0.9}$	δ_5	$A_4(0.5266)$	$A_3(0.5200)$	$A_1(0.4842)$	$A_2(0.4592)$	0.0066
$O_{\alpha=0.1}$	δ_3	$A_3(0.5239)$	$A_4(0.5174)$	$A_1(0.4813)$	$A_2(0.4713)$	0.0065
$O_{\alpha=-0.2}$	δ_3	$A_3(0.5268)$	$A_4(0.5207)$	$A_1(0.4788)$	$A_2(0.4724)$	0.0061
$O_{\alpha=0.9}$	δ_3	$A_3(0.5235)$	$A_4(0.5175)$	$A_1(0.4813)$	$A_2(0.4718)$	0.0060
$O_{\alpha=-0.3}$	δ_1	$A_3(0.5485)$	$A_4(0.5426)$	$A_2(0.4627)$	$A_1(0.4494)$	0.0059
$O_{\alpha=-0.5}$	δ_3	$A_3(0.5259)$	$A_4(0.5200)$	$A_1(0.4804)$	$A_2(0.4719)$	0.0059
$O_{\alpha=0.5}$	δ_3	$A_3(0.5252)$	$A_4(0.5193)$	$A_1(0.4799)$	$A_2(0.4724)$	0.0059
O_{RS}	δ_5	$A_3(0.5257)$	$A_4(0.5201)$	$A_1(0.4793)$	$A_2(0.4716)$	0.0056
T_P	δ_3	$A_3(0.5257)$	$A_4(0.5201)$	$A_1(0.4793)$	$A_2(0.4716)$	0.0056
$O_{\alpha=-0.6}$	δ_3	$A_3(0.5252)$	$A_4(0.5199)$	$A_1(0.4771)$	$A_2(0.4720)$	0.0054

Table 23: Rankings and differences $\Delta_{R1,R2}$ of the problem by using dC_F -integral with several combinations for F and δ functions.

F	δ	Ranked 1st	Ranked 2nd	Ranked 3rd	Ranked 4th	$\Delta_{R1,R2}$
$O_{\alpha=0.3}$	δ_3	$A_3(0.5249)$	$A_4(0.5196)$	$A_1(0.4788)$	$A_2(0.4720)$	0.0053
$O_{\alpha=-0.8}$	δ_3	$A_3(0.5241)$	$A_4(0.5190)$	$A_1(0.4792)$	$A_2(0.4697)$	0.0052
$O_{\alpha=0.5}$	δ_0	$A_3(0.5438)$	$A_4(0.5387)$	$A_1(0.4549)$	$A_2(0.4440)$	0.0052
$O_{\alpha=0.6}$	δ_3	$A_3(0.5245)$	$A_4(0.5196)$	$A_1(0.4804)$	$A_2(0.4717)$	0.0050
$O_{\alpha=1}$	δ_1	$A_4(0.5399)$	$A_3(0.5349)$	$A_2(0.4619)$	$A_1(0.4501)$	0.0050
GM	δ_5	$A_3(0.5257)$	$A_4(0.5208)$	$A_1(0.4796)$	$A_2(0.4692)$	0.0049
O_{mM}	δ_3	$A_3(0.5238)$	$A_4(0.5189)$	$A_1(0.4811)$	$A_2(0.4698)$	0.0049
C_F	δ_1	$A_4(0.5462)$	$A_3(0.5416)$	$A_2(0.4591)$	$A_1(0.4470)$	0.0047
$O_{\alpha=0.2}$	δ_3	$A_3(0.5248)$	$A_4(0.5202)$	$A_1(0.4777)$	$A_2(0.4721)$	0.0046
$O_{\alpha=0.7}$	δ_5	$A_4(0.5267)$	$A_3(0.5223)$	$A_1(0.4935)$	$A_2(0.4591)$	0.0044
O_{mM}	δ_1	$A_3(0.5490)$	$A_4(0.5449)$	$A_2(0.4591)$	$A_1(0.4468)$	0.0041
$O_{\alpha=0.6}$	δ_1	$A_4(0.5420)$	$A_3(0.5381)$	$A_2(0.4629)$	$A_1(0.4540)$	0.0039
$O_{\alpha=0.8}$	δ_3	$A_3(0.5235)$	$A_4(0.5197)$	$A_1(0.4769)$	$A_2(0.4721)$	0.0038
$O_{\alpha=0.6}$	δ_5	$A_4(0.5260)$	$A_3(0.5223)$	$A_1(0.4903)$	$A_2(0.4584)$	0.0037
$O_{\alpha=-0.5}$	δ_1	$A_3(0.5466)$	$A_4(0.5431)$	$A_2(0.4593)$	$A_1(0.4408)$	0.0035
$O_{\alpha=-0.6}$	δ_1	$A_3(0.5509)$	$A_4(0.5475)$	$A_2(0.4594)$	$A_1(0.4383)$	0.0035
$O_{\alpha=0.8}$	δ_5	$A_4(0.5241)$	$A_3(0.5206)$	$A_1(0.4894)$	$A_2(0.4584)$	0.0035
$O_{\alpha=0.1}$	δ_5	$A_4(0.5270)$	$A_3(0.5238)$	$A_1(0.4890)$	$A_2(0.4586)$	0.0033
C_L	δ_4	$A_4(0.5236)$	$A_3(0.5205)$	$A_1(0.4709)$	$A_2(0.4153)$	0.0031
O_{RS}	δ_4	$A_4(0.5623)$	$A_3(0.5592)$	$A_1(0.4523)$	$A_2(0.3981)$	0.0031
F_{BPC}	δ_5	$A_3(0.5269)$	$A_4(0.5239)$	$A_1(0.4941)$	$A_2(0.4596)$	0.0030
$O_{\alpha=0.5}$	δ_1	$A_4(0.5425)$	$A_3(0.5396)$	$A_2(0.4630)$	$A_1(0.4472)$	0.0029
C_L	δ_3	$A_4(0.5178)$	$A_3(0.5150)$	$A_1(0.4775)$	$A_2(0.4673)$	0.0028
$O_{\alpha=0.8}$	δ_1	$A_3(0.5432)$	$A_4(0.5404)$	$A_2(0.4638)$	$A_1(0.4602)$	0.0028
C_F	δ_3	$A_3(0.5229)$	$A_4(0.5202)$	$A_1(0.4766)$	$A_2(0.4700)$	0.0027
$O_{\alpha=-0.4}$	δ_3	$A_3(0.5242)$	$A_4(0.5215)$	$A_1(0.4761)$	$A_2(0.4718)$	0.0027
O_B	δ_3	$A_4(0.5212)$	$A_3(0.5185)$	$A_1(0.4758)$	$A_2(0.4682)$	0.0027
T_P	δ_1	$A_3(0.5454)$	$A_4(0.5429)$	$A_2(0.4575)$	$A_1(0.4486)$	0.0024
$O_{\alpha=0.4}$	δ_3	$A_3(0.5231)$	$A_4(0.5207)$	$A_1(0.4788)$	$A_2(0.4708)$	0.0023
$F_{BD1,q=0.1}$	δ_2	$A_4(0.5022)$	$A_1(0.5000)$	$A_3(0.5000)$	$A_2(0.4889)$	0.0022
$F_{BD1,q=0.2}$	δ_2	$A_4(0.5022)$	$A_1(0.5000)$	$A_3(0.5000)$	$A_2(0.4889)$	0.0022
$F_{BD1,q=0.3}$	δ_2	$A_4(0.5022)$	$A_1(0.5000)$	$A_3(0.5000)$	$A_2(0.4889)$	0.0022
$F_{BD1,q=0.4}$	δ_2	$A_4(0.5022)$	$A_1(0.5000)$	$A_3(0.5000)$	$A_2(0.4889)$	0.0022
$F_{BD1,q=0.5}$	δ_2	$A_4(0.5022)$	$A_1(0.5000)$	$A_3(0.5000)$	$A_2(0.4889)$	0.0022
$F_{BD1,q=0.6}$	δ_2	$A_4(0.5022)$	$A_1(0.5000)$	$A_3(0.5000)$	$A_2(0.4889)$	0.0022

Table 23: Rankings and differences $\Delta_{R1,R2}$ of the problem by using dC_F -integral with several combinations for F and δ functions.

F	δ	Ranked 1st	Ranked 2nd	Ranked 3rd	Ranked 4th	$\Delta_{R1,R2}$
$F_{BD1,q=0.7}$	δ_2	$A_4(0.5022)$	$A_1(0.5000)$	$A_3(0.5000)$	$A_2(0.4889)$	0.0022
$F_{BD1,q=0.8}$	δ_2	$A_4(0.5022)$	$A_1(0.5000)$	$A_3(0.5000)$	$A_2(0.4889)$	0.0022
$F_{BD1,q=0.9}$	δ_2	$A_4(0.5022)$	$A_1(0.5000)$	$A_3(0.5000)$	$A_2(0.4889)$	0.0022
$F_{BD1,q=1}$	δ_2	$A_4(0.5022)$	$A_1(0.5000)$	$A_3(0.5000)$	$A_2(0.4889)$	0.0022
C_L	δ_2	$A_4(0.5041)$	$A_3(0.5020)$	$A_1(0.4999)$	$A_2(0.4849)$	0.0021
F_{GL}	δ_0	$A_4(0.5023)$	$A_3(0.5003)$	$A_1(0.5000)$	$A_2(0.4885)$	0.0020
$O_{\alpha=-1}$	δ_1	$A_3(0.5479)$	$A_4(0.5460)$	$A_2(0.4636)$	$A_1(0.4463)$	0.0019
$O_{\alpha=0.3}$	δ_5	$A_4(0.5264)$	$A_3(0.5245)$	$A_1(0.4888)$	$A_2(0.4587)$	0.0019
$O_{\alpha=0.5}$	δ_5	$A_4(0.5259)$	$A_3(0.5240)$	$A_1(0.4933)$	$A_2(0.4587)$	0.0019
$O_{\alpha=-0.9}$	δ_5	$A_3(0.5269)$	$A_4(0.5251)$	$A_1(0.4918)$	$A_2(0.4598)$	0.0018
C_F	δ_5	$A_4(0.5265)$	$A_3(0.5249)$	$A_1(0.4917)$	$A_2(0.4583)$	0.0017
T_{NM}	δ_2	$A_4(0.5056)$	$A_3(0.5040)$	$A_1(0.4998)$	$A_2(0.4831)$	0.0017
$O_{\alpha=-1}$	δ_3	$A_3(0.5226)$	$A_4(0.5210)$	$A_1(0.4784)$	$A_2(0.4716)$	0.0016
$O_{\alpha=0.9}$	δ_2	$A_4(0.5038)$	$A_3(0.5022)$	$A_1(0.5000)$	$A_2(0.4849)$	0.0016
O_{mM}	δ_2	$A_3(0.5071)$	$A_4(0.5054)$	$A_1(0.4998)$	$A_2(0.4807)$	0.0016
O_B	δ_2	$A_4(0.5039)$	$A_3(0.5023)$	$A_1(0.5000)$	$A_2(0.4844)$	0.0016
$O_{\alpha=-1}$	δ_2	$A_3(0.5075)$	$A_4(0.5059)$	$A_1(0.4997)$	$A_2(0.4808)$	0.0015
O_{Div}	δ_2	$A_4(0.5035)$	$A_3(0.5020)$	$A_1(0.5000)$	$A_2(0.4854)$	0.0015
O_{mM}	δ_5	$A_3(0.5261)$	$A_4(0.5246)$	$A_1(0.4929)$	$A_2(0.4590)$	0.0015
F_{BPC}	δ_2	$A_3(0.5078)$	$A_4(0.5064)$	$A_1(0.4997)$	$A_2(0.4794)$	0.0014
$O_{\alpha=0.2}$	δ_5	$A_3(0.5270)$	$A_4(0.5255)$	$A_1(0.4963)$	$A_2(0.4594)$	0.0014
$O_{\alpha=0.7}$	δ_2	$A_4(0.5040)$	$A_3(0.5026)$	$A_1(0.4999)$	$A_2(0.4847)$	0.0014
$O_{\alpha=0.8}$	δ_2	$A_4(0.5039)$	$A_3(0.5025)$	$A_1(0.5000)$	$A_2(0.4845)$	0.0014
$O_{\alpha=-0.7}$	δ_5	$A_3(0.5258)$	$A_4(0.5245)$	$A_1(0.4905)$	$A_2(0.4599)$	0.0013
$O_{\alpha=1}$	δ_2	$A_4(0.5037)$	$A_3(0.5024)$	$A_1(0.5000)$	$A_2(0.4850)$	0.0013
T_{HP}	δ_2	$A_4(0.5033)$	$A_3(0.5020)$	$A_1(0.5000)$	$A_2(0.4859)$	0.0012
$O_{\alpha=-0.4}$	δ_1	$A_3(0.5445)$	$A_4(0.5434)$	$A_2(0.4637)$	$A_1(0.4525)$	0.0011
$O_{\alpha=-0.9}$	δ_2	$A_3(0.5063)$	$A_4(0.5052)$	$A_1(0.4998)$	$A_2(0.4813)$	0.0011
$O_{\alpha=0.7}$	δ_3	$A_3(0.5225)$	$A_4(0.5214)$	$A_1(0.4768)$	$A_2(0.4711)$	0.0011
F_{NA}	δ_2	$A_4(0.5031)$	$A_3(0.5020)$	$A_1(0.5000)$	$A_2(0.4864)$	0.0011
$O_{\alpha=-0.4}$	δ_5	$A_4(0.5259)$	$A_3(0.5249)$	$A_1(0.4936)$	$A_2(0.4589)$	0.0010
$O_{\alpha=-0.6}$	δ_5	$A_3(0.5258)$	$A_4(0.5247)$	$A_1(0.4933)$	$A_2(0.4591)$	0.0010
$O_{\alpha=0.2}$	δ_2	$A_4(0.5048)$	$A_3(0.5037)$	$A_1(0.4999)$	$A_2(0.4833)$	0.0010
T_M	δ_2	$A_4(0.5030)$	$A_3(0.5020)$	$A_1(0.5000)$	$A_2(0.4864)$	0.0010
F_{NA}	δ_3	$A_4(0.5185)$	$A_3(0.5175)$	$A_1(0.4775)$	$A_2(0.4667)$	0.0010

Table 23: Rankings and differences $\Delta_{R1,R2}$ of the problem by using dC_F -integral with several combinations for F and δ functions.

F	δ	Ranked 1st	Ranked 2nd	Ranked 3rd	Ranked 4th	$\Delta_{R1,R2}$
$O_{\alpha=-0.6}$	δ_2	$A_3(0.5056)$	$A_4(0.5047)$	$A_1(0.4998)$	$A_2(0.4827)$	0.0009
$O_{\alpha=0.1}$	δ_2	$A_4(0.5044)$	$A_3(0.5035)$	$A_1(0.4999)$	$A_2(0.4831)$	0.0009
$O_{\alpha=0.4}$	δ_2	$A_4(0.5041)$	$A_3(0.5032)$	$A_1(0.4999)$	$A_2(0.4838)$	0.0009
$O_{\alpha=0.5}$	δ_2	$A_4(0.5040)$	$A_3(0.5030)$	$A_1(0.4999)$	$A_2(0.4846)$	0.0009
C_F	δ_2	$A_3(0.5047)$	$A_4(0.5039)$	$A_1(0.4999)$	$A_2(0.4846)$	0.0008
$O_{\alpha=0.2}$	δ_1	$A_4(0.5456)$	$A_3(0.5448)$	$A_2(0.4590)$	$A_1(0.4566)$	0.0008
T_L	δ_2	$A_3(0.5075)$	$A_4(0.5067)$	$A_1(0.4996)$	$A_2(0.4772)$	0.0008
$O_{\alpha=-0.1}$	δ_2	$A_4(0.5048)$	$A_3(0.5041)$	$A_1(0.4998)$	$A_2(0.4829)$	0.0007
$O_{\alpha=0.6}$	δ_2	$A_4(0.5040)$	$A_3(0.5033)$	$A_1(0.4999)$	$A_2(0.4842)$	0.0007
$O_{\alpha=-0.2}$	δ_2	$A_4(0.5047)$	$A_3(0.5041)$	$A_1(0.4999)$	$A_2(0.4833)$	0.0006
$O_{\alpha=0.7}$	δ_1	$A_3(0.5413)$	$A_4(0.5407)$	$A_2(0.4638)$	$A_1(0.4548)$	0.0006
GM	δ_0	$A_4(0.5027)$	$A_3(0.5021)$	$A_1(0.5000)$	$A_2(0.4867)$	0.0006
$O_{\alpha=-0.5}$	δ_5	$A_3(0.5251)$	$A_4(0.5246)$	$A_1(0.4876)$	$A_2(0.4598)$	0.0005
$O_{\alpha=0.3}$	δ_2	$A_4(0.5041)$	$A_3(0.5036)$	$A_1(0.4999)$	$A_2(0.4845)$	0.0005
$O_{\alpha=0.3}$	δ_1	$A_4(0.5416)$	$A_3(0.5411)$	$A_2(0.4640)$	$A_1(0.4481)$	0.0005
$O_{\alpha=-0.7}$	δ_2	$A_4(0.5055)$	$A_3(0.5051)$	$A_1(0.4998)$	$A_2(0.4819)$	0.0004
$O_{\alpha=-0.8}$	δ_2	$A_3(0.5053)$	$A_4(0.5049)$	$A_1(0.4998)$	$A_2(0.4817)$	0.0004
$O_{\alpha=0.4}$	δ_5	$A_4(0.5251)$	$A_3(0.5248)$	$A_1(0.4929)$	$A_2(0.4589)$	0.0004
O_{Div}	δ_0	$A_3(0.5289)$	$A_4(0.5284)$	$A_1(0.4728)$	$A_2(0.4455)$	0.0004
$O_{\alpha=-0.1}$	δ_5	$A_4(0.5255)$	$A_3(0.5252)$	$A_1(0.4904)$	$A_2(0.4588)$	0.0003
$O_{\alpha=-0.3}$	δ_2	$A_3(0.5051)$	$A_4(0.5048)$	$A_1(0.4999)$	$A_2(0.4825)$	0.0003
$O_{\alpha=-0.5}$	δ_2	$A_4(0.5048)$	$A_3(0.5045)$	$A_1(0.4998)$	$A_2(0.4828)$	0.0003
$O_{\alpha=-0.3}$	δ_5	$A_4(0.5250)$	$A_3(0.5247)$	$A_1(0.4931)$	$A_2(0.4598)$	0.0002
$O_{\alpha=-0.4}$	δ_2	$A_4(0.5048)$	$A_3(0.5046)$	$A_1(0.4998)$	$A_2(0.4826)$	0.0002
$O_{\alpha=-0.2}$	δ_5	$A_4(0.5260)$	$A_3(0.5259)$	$A_1(0.4946)$	$A_2(0.4589)$	0.0001
$O_{\alpha=-0.8}$	δ_5	$A_3(0.5247)$	$A_4(0.5245)$	$A_1(0.4933)$	$A_2(0.4578)$	0.0001
$O_{\alpha=-1}$	δ_5	$A_3(0.5254)$	$A_4(0.5252)$	$A_1(0.4911)$	$A_2(0.4591)$	0.0001
$O_{\alpha=1}$	δ_3	$A_4(0.5205)$	$A_3(0.5204)$	$A_1(0.4765)$	$A_2(0.4696)$	0.0001
O_{RS}	δ_0	$A_4(0.5045)$	$A_3(0.5044)$	$A_1(0.4999)$	$A_2(0.4831)$	0.0001
T_P	δ_2	$A_4(0.5045)$	$A_3(0.5044)$	$A_1(0.4999)$	$A_2(0.4831)$	0.0001
T_P	δ_5	$A_4(0.5253)$	$A_3(0.5251)$	$A_1(0.4878)$	$A_2(0.4602)$	0.0001
F_{GL}	δ_2	$A_1(0.5000)$	$A_2(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	0.0000
F_{GL}	δ_3	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.5000)$	0.0000
GM	δ_2	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.5000)$	0.0000
GM	δ_3	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.4999)$	0.0000

Table 23: Rankings and differences $\Delta_{R1,R2}$ of the problem by using dC_F -integral with several combinations for F and δ functions.

F	δ	Ranked 1st	Ranked 2nd	Ranked 3rd	Ranked 4th	$\Delta_{R1,R2}$
HM	δ_2	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.4999)$	0.0000
O_{RS}	δ_2	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.4999)$	0.0000
O_{RS}	δ_3	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.4998)$	0.0000
Sin	δ_0	$A_1(0.5000)$	$A_2(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	0.0000
Sin	δ_2	$A_1(0.5000)$	$A_2(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	0.0000
Sin	δ_3	$A_1(0.5000)$	$A_2(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	0.0000
Sin	δ_4	$A_1(0.5000)$	$A_2(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	0.0000
Sin	δ_1	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.5000)$	0.0000
Sin	δ_5	$A_1(0.5000)$	$A_2(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	0.0000
F_{NA2}	δ_0	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.4999)$	0.0000
F_{NA2}	δ_2	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.5000)$	0.0000
F_{NA2}	δ_3	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.5000)$	0.0000
F_{NA2}	δ_4	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.4998)$	0.0000
F_{NA2}	δ_1	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.4998)$	0.0000
F_{NA2}	δ_5	$A_1(0.5000)$	$A_3(0.5000)$	$A_4(0.5000)$	$A_2(0.4999)$	0.0000

6 Conclusion

The discrete Choquet integral is an averaging aggregation function, which were generalized into a series of new integrals, like the C_T -integrals, CC -integrals, C_F -integrals and d-Choquet integral. They were applied in various problems and presented solid results when compared with the standard Choquet integral.

In this Master's thesis we presented two of those applications, the classification one, using FRBCS, and decision making, with the GMC-RTOPSIS. We applied the C_T -integral, which uses t-norms, in the GMC-RTOPSIS, by defining the C_T -separation measure, and concluded that it is a good alternative to the standard Choquet integral. Our method of comparison was the difference between the alternative ranked first and the one ranked second.

Afterwards, we defined the CC -separation measure, based on the CC -integral, that uses copula functions instead of the product in the expanded Choquet integral, and applied it in the GMC-RTOPSIS. The results were akin to the ones from the C_T -separation measure, by using the difference when comparing the results give by applying distinct copula functions.

In the sequence, we introduced a new generalization of the Choquet integral, based on the functions F and dissimilarities, called dC_F -integrals. We, then, proceeded to apply it in the FRBCS and in the GMC-RTOPSIS. In both of these applications, the dC_F -integral have presented a great performance.

Lastly we discussed briefly about the use of the difference in the GMC-RTOPSIS, and presented alternatives to this methodology. Finally, for future works we intend on testing on learning the parameters q and α of the F_{BD1} and O_α functions for the GMC-RTOPSIS method, and testing different methodologies for choosing the rank when considering multiple dC_F -integrals.

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A Mean and standard deviation values for all combinations of functions F and δ

Table 24: Results for state 1 (S1)

F	δ	A_1		A_2		A_3		A_4	
		mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
F_{BPC}	$\delta 0$	0.4716	0.0032	0.4606	0.0050	0.5558	0.0255	0.5360	0.0034
F_{BPC}	$\delta 2$	0.4999	0.0009	0.4667	0.0116	0.5145	0.0101	0.5134	0.0101
F_{BPC}	$\delta 3$	0.4828	0.0012	0.4788	0.0024	0.5283	0.0116	0.5195	0.0024
F_{BPC}	$\delta 4$	0.4589	0.0060	0.4336	0.0074	0.6025	0.0639	0.5619	0.0056
F_{BPC}	$\delta 1$	0.4134	0.0203	0.5110	0.0274	0.5390	0.0359	0.5214	0.0409
F_{BPC}	$\delta 5$	0.4683	0.0261	0.4792	0.0361	0.5065	0.0289	0.4985	0.0278
F_{GL}	$\delta 0$	0.5000	0.0003	0.4781	0.0150	0.5008	0.0014	0.5058	0.0088
F_{GL}	$\delta 2$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
F_{GL}	$\delta 3$	0.5000	0.0000	0.4999	0.0007	0.5000	0.0000	0.5000	0.0000
F_{GL}	$\delta 4$	0.4391	0.0324	0.4002	0.0407	0.5279	0.0611	0.5704	0.0432
F_{GL}	$\delta 1$	0.4393	0.0496	0.3950	0.0360	0.5373	0.0872	0.5903	0.0426
F_{GL}	$\delta 5$	0.4656	0.0234	0.4478	0.0174	0.5175	0.0406	0.5451	0.0205
C	$\delta 0$	0.4575	0.0048	0.4552	0.0053	0.5551	0.0222	0.5419	0.0050
C_F	$\delta 2$	0.5000	0.0005	0.4710	0.0110	0.5105	0.0073	0.5092	0.0089
C_F	$\delta 3$	0.4761	0.0023	0.4774	0.0024	0.5266	0.0125	0.5213	0.0024
C_F	$\delta 4$	0.4602	0.0141	0.4286	0.0122	0.6018	0.0762	0.5654	0.0098
C_F	$\delta 1$	0.4178	0.0256	0.5037	0.0279	0.5370	0.0784	0.5256	0.0402
C_F	$\delta 5$	0.4697	0.0260	0.4771	0.0360	0.5080	0.0292	0.5000	0.0276
C_L	$\delta 0$	0.4417	0.0091	0.4418	0.0123	0.5063	0.0157	0.5507	0.0112
C_L	$\delta 2$	0.5000	0.0004	0.4710	0.0108	0.5050	0.0064	0.5097	0.0089
C_L	$\delta 3$	0.4681	0.0046	0.4713	0.0041	0.5142	0.0089	0.5268	0.0042
C_L	$\delta 4$	0.4819	0.0325	0.4536	0.0245	0.5515	0.0390	0.5277	0.0182
C_L	$\delta 1$	0.3977	0.0350	0.4920	0.0304	0.5092	0.0563	0.5296	0.0415

Table 24: Results for state 1 (S1)

F	δ	A_1		A_2		A_3		A_4	
		mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
C_L	$\delta 5$	0.4568	0.0279	0.4703	0.0351	0.4993	0.0311	0.5088	0.0279
$O_{\alpha=-0.1}$	$\delta 0$	0.4754	0.0041	0.4595	0.0049	0.5614	0.0299	0.5380	0.0025
$O_{\alpha=-0.1}$	$\delta 2$	0.4999	0.0006	0.4692	0.0110	0.5081	0.0102	0.5108	0.0094
$O_{\alpha=-0.1}$	$\delta 3$	0.4798	0.0016	0.4767	0.0030	0.5280	0.0120	0.5212	0.0024
$O_{\alpha=-0.1}$	$\delta 4$	0.4587	0.0152	0.4246	0.0121	0.5974	0.0652	0.5669	0.0131
$O_{\alpha=-0.1}$	$\delta 1$	0.4172	0.0234	0.5055	0.0276	0.5401	0.0663	0.5248	0.0402
$O_{\alpha=-0.1}$	$\delta 5$	0.4660	0.0260	0.4787	0.0358	0.5064	0.0292	0.4996	0.0277
$O_{\alpha=-0.2}$	$\delta 0$	0.4588	0.0041	0.4572	0.0054	0.5530	0.0223	0.5394	0.0047
$O_{\alpha=-0.2}$	$\delta 2$	0.4999	0.0006	0.4702	0.0110	0.5083	0.0096	0.5102	0.0092
$O_{\alpha=-0.2}$	$\delta 3$	0.4811	0.0022	0.4776	0.0022	0.5290	0.0106	0.5211	0.0018
$O_{\alpha=-0.2}$	$\delta 4$	0.4530	0.0120	0.4219	0.0107	0.6037	0.0730	0.5729	0.0083
$O_{\alpha=-0.2}$	$\delta 1$	0.4168	0.0237	0.5043	0.0275	0.5402	0.0651	0.5254	0.0403
$O_{\alpha=-0.2}$	$\delta 5$	0.4693	0.0257	0.4788	0.0359	0.5101	0.0277	0.4991	0.0277
$O_{\alpha=-0.3}$	$\delta 0$	0.4606	0.0045	0.4585	0.0046	0.5573	0.0244	0.5391	0.0042
$O_{\alpha=-0.3}$	$\delta 2$	0.4999	0.0007	0.4689	0.0111	0.5095	0.0111	0.5111	0.0095
$O_{\alpha=-0.3}$	$\delta 3$	0.4775	0.0014	0.4777	0.0023	0.5274	0.0106	0.5209	0.0019
$O_{\alpha=-0.3}$	$\delta 4$	0.4659	0.0128	0.4287	0.0120	0.5989	0.0708	0.5654	0.0094
$O_{\alpha=-0.3}$	$\delta 1$	0.4280	0.0237	0.5132	0.0280	0.5454	0.0673	0.5174	0.0410
$O_{\alpha=-0.3}$	$\delta 5$	0.4710	0.0262	0.4783	0.0363	0.5071	0.0302	0.4981	0.0279
$O_{\alpha=-0.4}$	$\delta 0$	0.4596	0.0026	0.4560	0.0042	0.5502	0.0236	0.5419	0.0026
$O_{\alpha=-0.4}$	$\delta 2$	0.4999	0.0006	0.4696	0.0113	0.5085	0.0120	0.5107	0.0092
$O_{\alpha=-0.4}$	$\delta 3$	0.4757	0.0016	0.4753	0.0030	0.5255	0.0105	0.5226	0.0028
$O_{\alpha=-0.4}$	$\delta 4$	0.4349	0.0245	0.4207	0.0120	0.6069	0.0641	0.5725	0.0114
$O_{\alpha=-0.4}$	$\delta 1$	0.4184	0.0233	0.5080	0.0278	0.5381	0.0631	0.5228	0.0404
$O_{\alpha=-0.4}$	$\delta 5$	0.4679	0.0261	0.4760	0.0360	0.5039	0.0299	0.4996	0.0274
$O_{\alpha=-0.5}$	$\delta 0$	0.4597	0.0032	0.4600	0.0044	0.5555	0.0267	0.5376	0.0032
$O_{\alpha=-0.5}$	$\delta 2$	0.4999	0.0009	0.4696	0.0113	0.5088	0.0104	0.5106	0.0094
$O_{\alpha=-0.5}$	$\delta 3$	0.4832	0.0016	0.4772	0.0026	0.5291	0.0109	0.5210	0.0019
$O_{\alpha=-0.5}$	$\delta 4$	0.4739	0.0120	0.4350	0.0128	0.6075	0.0803	0.5588	0.0086
$O_{\alpha=-0.5}$	$\delta 1$	0.4242	0.0247	0.5115	0.0283	0.5477	0.0589	0.5171	0.0409
$O_{\alpha=-0.5}$	$\delta 5$	0.4686	0.0261	0.4770	0.0359	0.5046	0.0291	0.4985	0.0276
$O_{\alpha=-0.6}$	$\delta 0$	0.4596	0.0033	0.4582	0.0038	0.5539	0.0267	0.5397	0.0030
$O_{\alpha=-0.6}$	$\delta 2$	0.4999	0.0007	0.4699	0.0113	0.5105	0.0109	0.5104	0.0093
$O_{\alpha=-0.6}$	$\delta 3$	0.4768	0.0018	0.4775	0.0028	0.5253	0.0113	0.5206	0.0027

Table 24: Results for state 1 (S1)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
$O_{\alpha=-0.6}$	$\delta 4$	0.4424	0.0111	0.4234	0.0135	0.6020	0.0755	0.5683	0.0126
$O_{\alpha=-0.6}$	$\delta 1$	0.4133	0.0223	0.5066	0.0275	0.5433	0.0577	0.5240	0.0403
$O_{\alpha=-0.6}$	$\delta 5$	0.4701	0.0262	0.4767	0.0361	0.5068	0.0280	0.4985	0.0275
$O_{\alpha=-0.7}$	$\delta 0$	0.4692	0.0035	0.4569	0.0054	0.5540	0.0265	0.5388	0.0049
$O_{\alpha=-0.7}$	$\delta 2$	0.4999	0.0009	0.4683	0.0113	0.5095	0.0111	0.5115	0.0096
$O_{\alpha=-0.7}$	$\delta 3$	0.4803	0.0010	0.4810	0.0023	0.5278	0.0137	0.5171	0.0017
$O_{\alpha=-0.7}$	$\delta 4$	0.4536	0.0102	0.4242	0.0113	0.6131	0.0700	0.5701	0.0073
$O_{\alpha=-0.7}$	$\delta 1$	0.4270	0.0226	0.5132	0.0282	0.5460	0.0564	0.5177	0.0414
$O_{\alpha=-0.7}$	$\delta 5$	0.4709	0.0264	0.4780	0.0364	0.5078	0.0289	0.4977	0.0279
$O_{\alpha=-0.8}$	$\delta 0$	0.4744	0.0033	0.4645	0.0071	0.5552	0.0320	0.5302	0.0069
$O_{\alpha=-0.8}$	$\delta 2$	0.4999	0.0008	0.4695	0.0112	0.5097	0.0114	0.5109	0.0094
$O_{\alpha=-0.8}$	$\delta 3$	0.4805	0.0015	0.4767	0.0051	0.5255	0.0123	0.5192	0.0049
$O_{\alpha=-0.8}$	$\delta 4$	0.4339	0.0094	0.4290	0.0076	0.6080	0.0837	0.5673	0.0083
$O_{\alpha=-0.8}$	$\delta 1$	0.4164	0.0217	0.5099	0.0276	0.5382	0.0541	0.5218	0.0406
$O_{\alpha=-0.8}$	$\delta 5$	0.4684	0.0262	0.4763	0.0361	0.5060	0.0291	0.5001	0.0273
$O_{\alpha=-0.9}$	$\delta 0$	0.4612	0.0055	0.4563	0.0053	0.5520	0.0250	0.5409	0.0050
$O_{\alpha=-0.9}$	$\delta 2$	0.4999	0.0009	0.4690	0.0113	0.5114	0.0114	0.5114	0.0096
$O_{\alpha=-0.9}$	$\delta 3$	0.4850	0.0014	0.4785	0.0019	0.5264	0.0138	0.5201	0.0011
$O_{\alpha=-0.9}$	$\delta 4$	0.4540	0.0125	0.4228	0.0091	0.6092	0.0817	0.5727	0.0097
$O_{\alpha=-0.9}$	$\delta 1$	0.4140	0.0211	0.5092	0.0275	0.5386	0.0520	0.5219	0.0407
$O_{\alpha=-0.9}$	$\delta 5$	0.4703	0.0258	0.4788	0.0362	0.5076	0.0290	0.4979	0.0279
$O_{\alpha=-1}$	$\delta 0$	0.4755	0.0046	0.4654	0.0089	0.5588	0.0327	0.5267	0.0091
$O_{\alpha=-1}$	$\delta 2$	0.4999	0.0008	0.4678	0.0114	0.5139	0.0104	0.5123	0.0098
$O_{\alpha=-1}$	$\delta 3$	0.4796	0.0017	0.4778	0.0031	0.5227	0.0146	0.5198	0.0025
$O_{\alpha=-1}$	$\delta 4$	0.4555	0.0094	0.4334	0.0205	0.6083	0.0886	0.5524	0.0205
$O_{\alpha=-1}$	$\delta 1$	0.4170	0.0212	0.5114	0.0276	0.5414	0.0504	0.5204	0.0408
$O_{\alpha=-1}$	$\delta 5$	0.4685	0.0258	0.4770	0.0363	0.5046	0.0291	0.4995	0.0277
$O_{\alpha=0.1}$	$\delta 0$	0.4752	0.0060	0.4588	0.0052	0.5606	0.0253	0.5384	0.0050
$O_{\alpha=0.1}$	$\delta 2$	0.5000	0.0005	0.4700	0.0110	0.5074	0.0089	0.5102	0.0092
$O_{\alpha=0.1}$	$\delta 3$	0.4856	0.0024	0.4796	0.0050	0.5259	0.0135	0.5168	0.0053
$O_{\alpha=0.1}$	$\delta 4$	0.4741	0.0142	0.4418	0.0153	0.5962	0.0787	0.5487	0.0113
$O_{\alpha=0.1}$	$\delta 1$	0.4147	0.0247	0.5031	0.0277	0.5365	0.0744	0.5265	0.0401
$O_{\alpha=0.1}$	$\delta 5$	0.4691	0.0258	0.4774	0.0361	0.5066	0.0293	0.5002	0.0276
$O_{\alpha=0.2}$	$\delta 0$	0.4629	0.0044	0.4566	0.0050	0.5572	0.0214	0.5408	0.0036

Table 24: Results for state 1 (S1)

F	δ	A_1		A_2		A_3		A_4	
		mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
$O_{\alpha=0.2}$	$\delta 2$	0.4999	0.0006	0.4696	0.0110	0.5077	0.0093	0.5109	0.0093
$O_{\alpha=0.2}$	$\delta 3$	0.4767	0.0021	0.4764	0.0024	0.5264	0.0106	0.5222	0.0021
$O_{\alpha=0.2}$	$\delta 4$	0.4768	0.0146	0.4340	0.0112	0.6015	0.0608	0.5599	0.0101
$O_{\alpha=0.2}$	$\delta 1$	0.4283	0.0258	0.5075	0.0286	0.5431	0.0744	0.5217	0.0410
$O_{\alpha=0.2}$	$\delta 5$	0.4689	0.0259	0.4778	0.0359	0.5074	0.0296	0.4999	0.0277
$O_{\alpha=0.3}$	$\delta 0$	0.4618	0.0056	0.4585	0.0075	0.5556	0.0213	0.5366	0.0079
$O_{\alpha=0.3}$	$\delta 2$	0.5000	0.0005	0.4710	0.0110	0.5075	0.0091	0.5094	0.0089
$O_{\alpha=0.3}$	$\delta 3$	0.4802	0.0023	0.4769	0.0027	0.5295	0.0086	0.5216	0.0023
$O_{\alpha=0.3}$	$\delta 4$	0.4442	0.0286	0.4239	0.0139	0.5677	0.0782	0.5689	0.0154
$O_{\alpha=0.3}$	$\delta 1$	0.4299	0.0263	0.5108	0.0282	0.5399	0.0798	0.5188	0.0409
$O_{\alpha=0.3}$	$\delta 5$	0.4686	0.0257	0.4772	0.0358	0.5075	0.0291	0.5006	0.0274
$O_{\alpha=0.4}$	$\delta 0$	0.4722	0.0043	0.4578	0.0049	0.5505	0.0282	0.5396	0.0031
$O_{\alpha=0.4}$	$\delta 2$	0.5000	0.0005	0.4707	0.0109	0.5070	0.0084	0.5095	0.0091
$O_{\alpha=0.4}$	$\delta 3$	0.4819	0.0019	0.4764	0.0026	0.5274	0.0111	0.5220	0.0020
$O_{\alpha=0.4}$	$\delta 4$	0.4593	0.0159	0.4263	0.0143	0.5980	0.0677	0.5658	0.0130
$O_{\alpha=0.4}$	$\delta 1$	0.4169	0.0259	0.5024	0.0277	0.5300	0.0755	0.5269	0.0401
$O_{\alpha=0.4}$	$\delta 5$	0.4662	0.0254	0.4775	0.0356	0.5043	0.0301	0.5010	0.0275
$O_{\alpha=0.5}$	$\delta 0$	0.4527	0.0064	0.4550	0.0050	0.5512	0.0211	0.5425	0.0053
$O_{\alpha=0.5}$	$\delta 2$	0.5000	0.0006	0.4710	0.0111	0.5067	0.0084	0.5093	0.0089
$O_{\alpha=0.5}$	$\delta 3$	0.4815	0.0019	0.4783	0.0027	0.5280	0.0131	0.5200	0.0022
$O_{\alpha=0.5}$	$\delta 4$	0.4678	0.0145	0.4315	0.0156	0.5989	0.0621	0.5586	0.0145
$O_{\alpha=0.5}$	$\delta 1$	0.4268	0.0280	0.5081	0.0281	0.5379	0.0795	0.5216	0.0406
$O_{\alpha=0.5}$	$\delta 5$	0.4681	0.0258	0.4782	0.0357	0.5062	0.0302	0.5002	0.0276
$O_{\alpha=0.6}$	$\delta 0$	0.4703	0.0063	0.4565	0.0049	0.5572	0.0236	0.5411	0.0053
$O_{\alpha=0.6}$	$\delta 2$	0.5000	0.0005	0.4707	0.0110	0.5071	0.0086	0.5096	0.0091
$O_{\alpha=0.6}$	$\delta 3$	0.4844	0.0032	0.4775	0.0029	0.5304	0.0108	0.5210	0.0027
$O_{\alpha=0.6}$	$\delta 4$	0.4722	0.0144	0.4309	0.0137	0.5948	0.0706	0.5623	0.0097
$O_{\alpha=0.6}$	$\delta 1$	0.4421	0.0292	0.5161	0.0289	0.5456	0.0902	0.5132	0.0416
$O_{\alpha=0.6}$	$\delta 5$	0.4691	0.0261	0.4771	0.0360	0.5076	0.0293	0.4994	0.0278
$O_{\alpha=0.7}$	$\delta 0$	0.4773	0.0054	0.4622	0.0069	0.5547	0.0252	0.5335	0.0058
$O_{\alpha=0.7}$	$\delta 2$	0.5000	0.0004	0.4711	0.0109	0.5061	0.0076	0.5094	0.0089
$O_{\alpha=0.7}$	$\delta 3$	0.4785	0.0018	0.4762	0.0024	0.5287	0.0089	0.5224	0.0019
$O_{\alpha=0.7}$	$\delta 4$	0.4611	0.0148	0.4263	0.0102	0.5967	0.0663	0.5681	0.0108
$O_{\alpha=0.7}$	$\delta 1$	0.4407	0.0281	0.5125	0.0287	0.5448	0.0858	0.5168	0.0410

Table 24: Results for state 1 (S1)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
$O_{\alpha=0.7}$	$\delta 5$	0.4696	0.0260	0.4784	0.0358	0.5076	0.0294	0.4996	0.0277
$O_{\alpha=0.8}$	$\delta 0$	0.4748	0.0061	0.4590	0.0059	0.5569	0.0219	0.5378	0.0049
$O_{\alpha=0.8}$	$\delta 2$	0.5000	0.0005	0.4712	0.0109	0.5059	0.0074	0.5093	0.0090
$O_{\alpha=0.8}$	$\delta 3$	0.4744	0.0028	0.4762	0.0023	0.5260	0.0094	0.5224	0.0024
$O_{\alpha=0.8}$	$\delta 4$	0.4666	0.0151	0.4382	0.0228	0.5792	0.0735	0.5467	0.0259
$O_{\alpha=0.8}$	$\delta 1$	0.4383	0.0287	0.5129	0.0286	0.5389	0.0877	0.5167	0.0412
$O_{\alpha=0.8}$	$\delta 5$	0.4679	0.0257	0.4768	0.0356	0.5068	0.0295	0.5014	0.0273
$O_{\alpha=0.9}$	$\delta 0$	0.4697	0.0046	0.4571	0.0050	0.5480	0.0243	0.5405	0.0038
$O_{\alpha=0.9}$	$\delta 2$	0.5000	0.0004	0.4715	0.0110	0.5055	0.0070	0.5090	0.0089
$O_{\alpha=0.9}$	$\delta 3$	0.4845	0.0024	0.4779	0.0057	0.5257	0.0122	0.5177	0.0057
$O_{\alpha=0.9}$	$\delta 4$	0.4587	0.0150	0.4244	0.0104	0.5923	0.0568	0.5698	0.0114
$O_{\alpha=0.9}$	$\delta 1$	0.4248	0.0281	0.5030	0.0281	0.5323	0.0855	0.5257	0.0402
$O_{\alpha=0.9}$	$\delta 5$	0.4615	0.0249	0.4780	0.0356	0.5055	0.0293	0.5006	0.0276
$O_{\alpha=1}$	$\delta 0$	0.4766	0.0063	0.4599	0.0059	0.5501	0.0269	0.5371	0.0049
$O_{\alpha=1}$	$\delta 2$	0.5000	0.0004	0.4715	0.0108	0.5058	0.0073	0.5089	0.0089
$O_{\alpha=1}$	$\delta 3$	0.4703	0.0024	0.4722	0.0021	0.5200	0.0094	0.5267	0.0022
$O_{\alpha=1}$	$\delta 4$	0.4580	0.0178	0.4237	0.0127	0.5899	0.0614	0.5698	0.0139
$O_{\alpha=1}$	$\delta 1$	0.4271	0.0367	0.5111	0.0285	0.5332	0.0932	0.5173	0.0413
$O_{\alpha=1}$	$\delta 5$	0.4664	0.0254	0.4777	0.0354	0.5045	0.0288	0.5011	0.0275
O_{Div}	$\delta 0$	0.4739	0.0070	0.4583	0.0071	0.5375	0.0146	0.5376	0.0062
O_{Div}	$\delta 2$	0.5000	0.0004	0.4721	0.0108	0.5051	0.0068	0.5085	0.0086
O_{Div}	$\delta 3$	0.4676	0.0036	0.4700	0.0034	0.5122	0.0072	0.5282	0.0036
O_{Div}	$\delta 4$	0.4706	0.0251	0.4403	0.0224	0.5475	0.0504	0.5421	0.0204
O_{Div}	$\delta 1$	0.4295	0.0388	0.4915	0.0303	0.5525	0.0803	0.5302	0.0417
O_{Div}	$\delta 5$	0.4648	0.0281	0.4709	0.0351	0.5150	0.0349	0.5084	0.0276
GM	$\delta 0$	0.5000	0.0003	0.4746	0.0128	0.5052	0.0068	0.5068	0.0090
GM	$\delta 2$	0.5000	0.0000	0.4999	0.0010	0.5000	0.0000	0.5000	0.0000
GM	$\delta 3$	0.5000	0.0000	0.4997	0.0022	0.5000	0.0000	0.5000	0.0000
GM	$\delta 4$	0.4241	0.0128	0.3843	0.0150	0.5314	0.0274	0.5959	0.0181
GM	$\delta 1$	0.4767	0.0162	0.4499	0.0112	0.5608	0.0430	0.5448	0.0129
GM	$\delta 5$	0.4778	0.0063	0.4742	0.0053	0.5277	0.0157	0.5232	0.0058
HM	$\delta 0$	0.4425	0.0075	0.4354	0.0077	0.5232	0.0158	0.5565	0.0090
HM	$\delta 2$	0.5000	0.0000	0.4997	0.0022	0.5000	0.0000	0.5000	0.0000
HM	$\delta 3$	0.4928	0.0030	0.4836	0.0044	0.4972	0.0115	0.5109	0.0044

Table 24: Results for state 1 (S1)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
HM	$\delta 4$	0.3947	0.0195	0.3965	0.0176	0.5314	0.0444	0.5926	0.0193
HM	$\delta 1$	0.4100	0.0443	0.4714	0.0302	0.5295	0.0838	0.5459	0.0407
HM	$\delta 5$	0.4600	0.0278	0.4724	0.0291	0.5085	0.0390	0.5128	0.0257
O_{mM}	$\delta 0$	0.4718	0.0035	0.4605	0.0036	0.5601	0.0243	0.5372	0.0032
O_{mM}	$\delta 2$	0.4999	0.0009	0.4689	0.0114	0.5118	0.0133	0.5115	0.0096
O_{mM}	$\delta 3$	0.4842	0.0015	0.4787	0.0036	0.5261	0.0140	0.5187	0.0028
O_{mM}	$\delta 4$	0.4476	0.0093	0.4261	0.0122	0.5979	0.0814	0.5660	0.0111
O_{mM}	$\delta 1$	0.4175	0.0213	0.5113	0.0279	0.5306	0.0534	0.5197	0.0410
O_{mM}	$\delta 5$	0.4684	0.0259	0.4785	0.0360	0.5052	0.0291	0.4979	0.0276
O_B	$\delta 0$	0.4738	0.0058	0.4648	0.0106	0.5463	0.0246	0.5278	0.0105
O_B	$\delta 2$	0.5000	0.0004	0.4710	0.0108	0.5055	0.0071	0.5093	0.0089
O_B	$\delta 3$	0.4694	0.0027	0.4715	0.0024	0.5181	0.0086	0.5273	0.0026
O_B	$\delta 4$	0.4688	0.0190	0.4307	0.0152	0.5851	0.0589	0.5612	0.0140
O_B	$\delta 1$	0.4222	0.0321	0.4979	0.0283	0.5382	0.0847	0.5286	0.0402
O_B	$\delta 5$	0.4664	0.0267	0.4755	0.0354	0.5078	0.0313	0.5029	0.0274
O_{RS}	$\delta 0$	0.5000	0.0006	0.4695	0.0111	0.5085	0.0101	0.5105	0.0093
O_{RS}	$\delta 2$	0.5000	0.0000	0.4997	0.0027	0.5000	0.0000	0.5000	0.0000
O_{RS}	$\delta 3$	0.5000	0.0000	0.4996	0.0031	0.5000	0.0000	0.5000	0.0000
O_{RS}	$\delta 4$	0.4443	0.0121	0.3824	0.0115	0.5633	0.0234	0.5891	0.0175
O_{RS}	$\delta 1$	0.4687	0.0040	0.4569	0.0060	0.5566	0.0250	0.5392	0.0048
O_{RS}	$\delta 5$	0.4823	0.0029	0.4781	0.0030	0.5281	0.0118	0.5198	0.0034
Sin	$\delta 0$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
Sin	$\delta 2$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
Sin	$\delta 3$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
Sin	$\delta 4$	0.5000	0.0000	0.5000	0.0003	0.5000	0.0000	0.5000	0.0000
Sin	$\delta 1$	0.5000	0.0000	0.4999	0.0006	0.5000	0.0000	0.5000	0.0000
Sin	$\delta 5$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
T_P	$\delta 0$	0.4687	0.0040	0.4569	0.0060	0.5566	0.0250	0.5392	0.0048
T_P	$\delta 2$	0.5000	0.0006	0.4695	0.0111	0.5085	0.0101	0.5105	0.0093
T_P	$\delta 3$	0.4823	0.0029	0.4781	0.0030	0.5281	0.0118	0.5198	0.0034
T_P	$\delta 4$	0.4567	0.0120	0.4270	0.0093	0.5976	0.0655	0.5674	0.0098
T_P	$\delta 1$	0.4212	0.0242	0.5092	0.0281	0.5387	0.0745	0.5205	0.0408
T_P	$\delta 5$	0.4694	0.0257	0.4790	0.0360	0.5076	0.0298	0.4985	0.0279
T_{DP}	$\delta 0$	0.0000	0.0000	0.0000	0.0000	0.8647	0.2518	0.0000	0.0000

Table 24: Results for state 1 (S1)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
T_{DP}	$\delta 2$	0.0000	0.0000	0.0000	0.0000	0.5322	0.2603	0.0000	0.0000
T_{DP}	$\delta 3$	0.0000	0.0000	0.0000	0.0000	0.4127	0.0752	0.0000	0.0000
T_{DP}	$\delta 4$	0.0000	0.0000	0.0000	0.0000	0.9031	0.2418	0.0000	0.0000
T_{DP}	$\delta 1$	0.0000	0.0000	0.0000	0.0000	0.9023	0.2433	0.0000	0.0000
T_{DP}	$\delta 5$	0.0000	0.0000	0.0000	0.0000	0.3616	0.0849	0.0000	0.0000
T_{HP}	$\delta 0$	0.4455	0.0140	0.4436	0.0085	0.5209	0.0162	0.5515	0.0091
T_{HP}	$\delta 2$	0.5000	0.0003	0.4726	0.0107	0.5051	0.0069	0.5080	0.0086
T_{HP}	$\delta 3$	0.4830	0.0049	0.4745	0.0046	0.5278	0.0098	0.5227	0.0046
T_{HP}	$\delta 4$	0.3891	0.0225	0.3974	0.0190	0.5218	0.0430	0.5937	0.0208
T_{HP}	$\delta 1$	0.4103	0.0452	0.4753	0.0312	0.5295	0.0844	0.5436	0.0422
T_{HP}	$\delta 5$	0.4606	0.0305	0.4686	0.0349	0.5076	0.0395	0.5113	0.0278
T_L	$\delta 0$	0.4601	0.0029	0.4693	0.0025	0.5618	0.0090	0.5301	0.0034
T_L	$\delta 2$	0.4998	0.0013	0.4671	0.0121	0.5136	0.0112	0.5135	0.0101
T_L	$\delta 3$	0.4816	0.0017	0.4861	0.0025	0.5330	0.0057	0.5129	0.0015
T_L	$\delta 4$	0.5187	0.0065	0.4674	0.0084	0.6852	0.0106	0.5279	0.0100
T_L	$\delta 1$	0.4857	0.0285	0.5559	0.0328	0.6568	0.0437	0.4792	0.0485
T_L	$\delta 5$	0.4876	0.0291	0.4864	0.0377	0.5263	0.0314	0.4889	0.0297
T_M	$\delta 0$	0.4463	0.0158	0.4465	0.0090	0.5147	0.0164	0.5490	0.0095
T_M	$\delta 2$	0.5000	0.0003	0.4731	0.0108	0.5050	0.0066	0.5076	0.0085
T_M	$\delta 3$	0.4576	0.0082	0.4604	0.0076	0.4971	0.0104	0.5361	0.0082
T_M	$\delta 4$	0.3877	0.0237	0.3980	0.0201	0.4957	0.0405	0.5917	0.0223
T_M	$\delta 1$	0.3831	0.0478	0.4572	0.0335	0.5392	0.0781	0.5582	0.0438
T_M	$\delta 5$	0.4490	0.0305	0.4598	0.0341	0.5079	0.0411	0.5225	0.0280
T_{NM}	$\delta 0$	0.4626	0.0029	0.4716	0.0015	0.5699	0.0231	0.5303	0.0029
T_{NM}	$\delta 2$	0.4999	0.0010	0.4689	0.0118	0.5077	0.0109	0.5123	0.0100
T_{NM}	$\delta 3$	0.4817	0.0017	0.4923	0.0026	0.5363	0.0111	0.5078	0.0019
T_{NM}	$\delta 4$	0.4695	0.0480	0.4526	0.0154	0.6431	0.0409	0.5403	0.0141
T_{NM}	$\delta 1$	0.4287	0.0252	0.5498	0.0314	0.6106	0.0264	0.4861	0.0473
T_{NM}	$\delta 5$	0.4885	0.0288	0.4874	0.0375	0.5255	0.0310	0.4880	0.0292
$F_{BD1,q=0.1}$	$\delta 0$	0.4186	0.0596	0.3754	0.0437	0.5218	0.0947	0.6074	0.0516
$F_{BD1,q=0.1}$	$\delta 2$	0.5000	0.0003	0.4787	0.0153	0.5000	0.0000	0.5056	0.0088
$F_{BD1,q=0.1}$	$\delta 3$	0.4587	0.0302	0.4364	0.0224	0.5112	0.0482	0.5547	0.0265
$F_{BD1,q=0.1}$	$\delta 4$	0.3458	0.1121	0.2691	0.0785	0.5396	0.1776	0.7004	0.0929
$F_{BD1,q=0.1}$	$\delta 1$	0.3984	0.1315	0.3661	0.0748	0.5236	0.2123	0.6295	0.0941

Table 24: Results for state 1 (S1)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
$F_{BD1,q=0.1}$	$\delta 5$	0.4662	0.0600	0.4515	0.0418	0.4984	0.1061	0.5296	0.0442
$F_{BD1,q=0.2}$	$\delta 0$	0.4186	0.0596	0.3754	0.0437	0.5218	0.0947	0.6074	0.0516
$F_{BD1,q=0.2}$	$\delta 2$	0.5000	0.0003	0.4787	0.0153	0.5000	0.0000	0.5056	0.0088
$F_{BD1,q=0.2}$	$\delta 3$	0.4587	0.0302	0.4364	0.0224	0.5112	0.0482	0.5547	0.0265
$F_{BD1,q=0.2}$	$\delta 4$	0.3443	0.1091	0.2681	0.0758	0.5403	0.1737	0.7019	0.0902
$F_{BD1,q=0.2}$	$\delta 1$	0.3984	0.1315	0.3661	0.0748	0.5236	0.2123	0.6295	0.0941
$F_{BD1,q=0.2}$	$\delta 5$	0.4662	0.0600	0.4515	0.0418	0.4984	0.1061	0.5296	0.0442
$F_{BD1,q=0.3}$	$\delta 0$	0.4183	0.0589	0.3753	0.0434	0.5225	0.0937	0.6076	0.0513
$F_{BD1,q=0.3}$	$\delta 2$	0.5000	0.0003	0.4787	0.0153	0.5000	0.0000	0.5056	0.0088
$F_{BD1,q=0.3}$	$\delta 3$	0.4587	0.0302	0.4364	0.0224	0.5112	0.0482	0.5547	0.0265
$F_{BD1,q=0.3}$	$\delta 4$	0.3424	0.1052	0.2692	0.0737	0.5410	0.1683	0.7008	0.0880
$F_{BD1,q=0.3}$	$\delta 1$	0.3984	0.1315	0.3661	0.0748	0.5236	0.2123	0.6295	0.0941
$F_{BD1,q=0.3}$	$\delta 5$	0.4662	0.0600	0.4515	0.0418	0.4984	0.1061	0.5296	0.0442
$F_{BD1,q=0.4}$	$\delta 0$	0.4175	0.0574	0.3749	0.0422	0.5236	0.0919	0.6082	0.0501
$F_{BD1,q=0.4}$	$\delta 2$	0.5000	0.0003	0.4787	0.0153	0.5000	0.0000	0.5056	0.0088
$F_{BD1,q=0.4}$	$\delta 3$	0.4587	0.0302	0.4364	0.0224	0.5112	0.0482	0.5547	0.0265
$F_{BD1,q=0.4}$	$\delta 4$	0.3405	0.1016	0.2746	0.0738	0.5417	0.1627	0.6945	0.0882
$F_{BD1,q=0.4}$	$\delta 1$	0.3984	0.1315	0.3661	0.0748	0.5236	0.2123	0.6295	0.0941
$F_{BD1,q=0.4}$	$\delta 5$	0.4662	0.0600	0.4515	0.0418	0.4984	0.1061	0.5296	0.0442
$F_{BD1,q=0.5}$	$\delta 0$	0.4168	0.0559	0.3744	0.0410	0.5244	0.0897	0.6090	0.0488
$F_{BD1,q=0.5}$	$\delta 2$	0.5000	0.0003	0.4787	0.0153	0.5000	0.0000	0.5056	0.0088
$F_{BD1,q=0.5}$	$\delta 3$	0.4587	0.0302	0.4364	0.0224	0.5112	0.0482	0.5547	0.0265
$F_{BD1,q=0.5}$	$\delta 4$	0.3388	0.0986	0.2788	0.0735	0.5424	0.1578	0.6897	0.0879
$F_{BD1,q=0.5}$	$\delta 1$	0.3984	0.1315	0.3661	0.0748	0.5236	0.2123	0.6295	0.0941
$F_{BD1,q=0.5}$	$\delta 5$	0.4662	0.0600	0.4515	0.0418	0.4984	0.1061	0.5296	0.0442
$F_{BD1,q=0.6}$	$\delta 0$	0.4161	0.0547	0.3739	0.0399	0.5249	0.0877	0.6097	0.0477
$F_{BD1,q=0.6}$	$\delta 2$	0.5000	0.0003	0.4787	0.0153	0.5000	0.0000	0.5056	0.0088
$F_{BD1,q=0.6}$	$\delta 3$	0.4587	0.0302	0.4364	0.0224	0.5112	0.0482	0.5547	0.0265
$F_{BD1,q=0.6}$	$\delta 4$	0.3374	0.0961	0.2820	0.0731	0.5430	0.1539	0.6863	0.0875
$F_{BD1,q=0.6}$	$\delta 1$	0.3984	0.1315	0.3661	0.0748	0.5236	0.2123	0.6295	0.0941
$F_{BD1,q=0.6}$	$\delta 5$	0.4662	0.0600	0.4515	0.0418	0.4984	0.1061	0.5296	0.0442
$F_{BD1,q=0.7}$	$\delta 0$	0.4156	0.0538	0.3735	0.0391	0.5253	0.0861	0.6102	0.0467
$F_{BD1,q=0.7}$	$\delta 2$	0.5000	0.0003	0.4787	0.0153	0.5000	0.0000	0.5056	0.0088
$F_{BD1,q=0.7}$	$\delta 3$	0.4587	0.0302	0.4364	0.0224	0.5112	0.0482	0.5547	0.0265

Table 24: Results for state 1 (S1)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
$F_{BD1,q=0.7}$	$\delta 4$	0.3364	0.0942	0.2844	0.0728	0.5435	0.1508	0.6838	0.0872
$F_{BD1,q=0.7}$	$\delta 1$	0.3984	0.1315	0.3661	0.0748	0.5236	0.2123	0.6295	0.0941
$F_{BD1,q=0.7}$	$\delta 5$	0.4662	0.0600	0.4515	0.0418	0.4984	0.1061	0.5296	0.0442
$F_{BD1,q=0.8}$	$\delta 0$	0.4152	0.0531	0.3733	0.0384	0.5256	0.0849	0.6107	0.0461
$F_{BD1,q=0.8}$	$\delta 2$	0.5000	0.0003	0.4787	0.0153	0.5000	0.0000	0.5056	0.0088
$F_{BD1,q=0.8}$	$\delta 3$	0.4587	0.0302	0.4364	0.0224	0.5112	0.0482	0.5547	0.0265
$F_{BD1,q=0.8}$	$\delta 4$	0.3357	0.0928	0.2860	0.0725	0.5439	0.1486	0.6821	0.0868
$F_{BD1,q=0.8}$	$\delta 1$	0.3984	0.1315	0.3661	0.0748	0.5236	0.2122	0.6295	0.0941
$F_{BD1,q=0.8}$	$\delta 5$	0.4662	0.0600	0.4515	0.0418	0.4984	0.1061	0.5296	0.0442
$F_{BD1,q=0.9}$	$\delta 0$	0.4149	0.0526	0.3731	0.0380	0.5258	0.0840	0.6110	0.0456
$F_{BD1,q=0.9}$	$\delta 2$	0.5000	0.0003	0.4787	0.0153	0.5000	0.0000	0.5056	0.0088
$F_{BD1,q=0.9}$	$\delta 3$	0.4587	0.0302	0.4364	0.0224	0.5112	0.0482	0.5547	0.0265
$F_{BD1,q=0.9}$	$\delta 4$	0.3351	0.0918	0.2872	0.0722	0.5442	0.1469	0.6810	0.0866
$F_{BD1,q=0.9}$	$\delta 1$	0.3984	0.1315	0.3661	0.0748	0.5237	0.2121	0.6295	0.0941
$F_{BD1,q=0.9}$	$\delta 5$	0.4662	0.0600	0.4515	0.0418	0.4984	0.1061	0.5296	0.0442
$F_{BD1,q=1}$	$\delta 0$	0.4147	0.0522	0.3729	0.0377	0.5260	0.0834	0.6112	0.0452
$F_{BD1,q=1}$	$\delta 2$	0.5000	0.0003	0.4787	0.0153	0.5000	0.0000	0.5056	0.0088
$F_{BD1,q=1}$	$\delta 3$	0.4587	0.0302	0.4364	0.0224	0.5112	0.0482	0.5547	0.0265
$F_{BD1,q=1}$	$\delta 4$	0.3347	0.0911	0.2880	0.0721	0.5444	0.1458	0.6802	0.0864
$F_{BD1,q=1}$	$\delta 1$	0.3984	0.1315	0.3661	0.0748	0.5237	0.2121	0.6295	0.0941
$F_{BD1,q=1}$	$\delta 5$	0.4662	0.0600	0.4515	0.0418	0.4984	0.1061	0.5296	0.0442
F_{NA2}	$\delta 0$	0.5000	0.0000	0.4997	0.0039	0.5000	0.0000	0.5000	0.0000
F_{NA2}	$\delta 2$	0.5000	0.0000	0.4999	0.0016	0.5000	0.0000	0.5000	0.0000
F_{NA2}	$\delta 3$	0.5000	0.0000	0.4999	0.0016	0.5000	0.0000	0.5000	0.0000
F_{NA2}	$\delta 4$	0.5000	0.0000	0.4996	0.0051	0.5000	0.0000	0.5000	0.0000
F_{NA2}	$\delta 1$	0.5000	0.0000	0.4995	0.0056	0.5000	0.0000	0.5000	0.0000
F_{NA2}	$\delta 5$	0.5000	0.0000	0.4997	0.0039	0.5000	0.0000	0.5000	0.0000
F_{NA}	$\delta 0$	0.4354	0.0093	0.4049	0.0367	0.5077	0.0181	0.5719	0.0332
F_{NA}	$\delta 2$	0.5000	0.0003	0.4731	0.0108	0.5050	0.0065	0.5077	0.0085
F_{NA}	$\delta 3$	0.4669	0.0053	0.4699	0.0046	0.5169	0.0089	0.5277	0.0051
F_{NA}	$\delta 4$	0.4438	0.0330	0.4302	0.0225	0.5310	0.0452	0.5543	0.0223
F_{NA}	$\delta 1$	0.3891	0.0476	0.4631	0.0331	0.5236	0.0801	0.5532	0.0436
F_{NA}	$\delta 5$	0.4495	0.0298	0.4605	0.0339	0.5020	0.0413	0.5221	0.0276

Table 25: Results for state 2 (S2)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
F_{BPC}	$\delta 0$	0.4541	0.0075	0.4365	0.0077	0.5558	0.0161	0.5479	0.0071
F_{BPC}	$\delta 2$	0.4996	0.0018	0.4879	0.0082	0.5033	0.0047	0.5017	0.0038
F_{BPC}	$\delta 3$	0.4817	0.0062	0.4677	0.0033	0.5271	0.0112	0.5196	0.0055
F_{BPC}	$\delta 4$	0.4443	0.0327	0.3774	0.0124	0.5856	0.0556	0.5646	0.0291
F_{BPC}	$\delta 1$	0.4743	0.0396	0.4369	0.0342	0.5659	0.0381	0.5548	0.0421
F_{BPC}	$\delta 5$	0.5113	0.0324	0.4466	0.0443	0.5405	0.0341	0.5408	0.0442
F_{GL}	$\delta 0$	0.5000	0.0000	0.4955	0.0059	0.5000	0.0000	0.5000	0.0000
F_{GL}	$\delta 2$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
F_{GL}	$\delta 3$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
F_{GL}	$\delta 4$	0.4753	0.0161	0.4281	0.0326	0.5279	0.0546	0.5220	0.0177
F_{GL}	$\delta 1$	0.4694	0.0371	0.3917	0.0361	0.5389	0.0855	0.5693	0.0310
F_{GL}	$\delta 5$	0.4812	0.0174	0.4463	0.0176	0.5185	0.0402	0.5337	0.0148
C_F	$\delta 0$	0.4589	0.0151	0.4392	0.0061	0.5468	0.0249	0.5354	0.0132
C_F	$\delta 2$	0.4999	0.0007	0.4936	0.0068	0.5009	0.0017	0.5004	0.0016
C_F	$\delta 3$	0.4769	0.0096	0.4650	0.0023	0.5205	0.0100	0.5195	0.0088
C_F	$\delta 4$	0.4326	0.0396	0.3898	0.0201	0.5895	0.0639	0.5577	0.0355
C_F	$\delta 1$	0.4664	0.0427	0.4294	0.0353	0.5446	0.0843	0.5600	0.0408
C_F	$\delta 5$	0.5064	0.0309	0.4457	0.0439	0.5361	0.0343	0.5442	0.0422
C_L	$\delta 0$	0.4708	0.0185	0.4358	0.0083	0.5082	0.0170	0.5195	0.0163
C_L	$\delta 2$	0.4999	0.0006	0.4941	0.0061	0.5000	0.0000	0.5003	0.0013
C_L	$\delta 3$	0.4838	0.0095	0.4647	0.0040	0.5155	0.0101	0.5118	0.0080
C_L	$\delta 4$	0.4635	0.0241	0.3897	0.0119	0.4998	0.0306	0.5208	0.0231
C_L	$\delta 1$	0.4883	0.0262	0.4151	0.0350	0.5137	0.0577	0.5382	0.0332
C_L	$\delta 5$	0.4977	0.0268	0.4364	0.0416	0.5113	0.0349	0.5478	0.0383
$O_{\alpha=-0.1}$	$\delta 0$	0.4613	0.0159	0.4377	0.0083	0.5442	0.0267	0.5370	0.0140
$O_{\alpha=-0.1}$	$\delta 2$	0.4998	0.0010	0.4920	0.0071	0.5015	0.0026	0.5008	0.0024
$O_{\alpha=-0.1}$	$\delta 3$	0.4795	0.0076	0.4677	0.0044	0.5251	0.0108	0.5179	0.0068
$O_{\alpha=-0.1}$	$\delta 4$	0.4158	0.0280	0.3838	0.0192	0.5831	0.0592	0.5752	0.0255
$O_{\alpha=-0.1}$	$\delta 1$	0.4599	0.0425	0.4295	0.0331	0.5414	0.0715	0.5644	0.0410
$O_{\alpha=-0.1}$	$\delta 5$	0.5066	0.0326	0.4456	0.0439	0.5378	0.0341	0.5428	0.0430
$O_{\alpha=-0.2}$	$\delta 0$	0.4586	0.0153	0.4356	0.0091	0.5450	0.0237	0.5362	0.0136
$O_{\alpha=-0.2}$	$\delta 2$	0.4999	0.0010	0.4921	0.0072	0.5013	0.0022	0.5010	0.0025
$O_{\alpha=-0.2}$	$\delta 3$	0.4773	0.0075	0.4689	0.0041	0.5254	0.0100	0.5205	0.0060
$O_{\alpha=-0.2}$	$\delta 4$	0.4229	0.0349	0.3855	0.0107	0.5836	0.0633	0.5669	0.0307

Table 25: Results for state 2 (S2)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
$O_{\alpha=-0.2}$	$\delta 1$	0.4547	0.0417	0.4303	0.0335	0.5436	0.0678	0.5648	0.0407
$O_{\alpha=-0.2}$	$\delta 5$	0.5114	0.0317	0.4457	0.0437	0.5365	0.0347	0.5440	0.0424
$O_{\alpha=-0.3}$	$\delta 0$	0.4638	0.0155	0.4353	0.0085	0.5500	0.0223	0.5318	0.0139
$O_{\alpha=-0.3}$	$\delta 2$	0.4999	0.0009	0.4915	0.0072	0.5022	0.0035	0.5006	0.0021
$O_{\alpha=-0.3}$	$\delta 3$	0.4810	0.0083	0.4703	0.0052	0.5265	0.0115	0.5162	0.0076
$O_{\alpha=-0.3}$	$\delta 4$	0.4251	0.0320	0.3812	0.0204	0.5881	0.0579	0.5685	0.0285
$O_{\alpha=-0.3}$	$\delta 1$	0.4637	0.0438	0.4291	0.0354	0.5506	0.0710	0.5594	0.0411
$O_{\alpha=-0.3}$	$\delta 5$	0.5078	0.0319	0.4474	0.0434	0.5365	0.0334	0.5429	0.0428
$O_{\alpha=-0.4}$	$\delta 0$	0.4589	0.0128	0.4396	0.0091	0.5500	0.0262	0.5367	0.0121
$O_{\alpha=-0.4}$	$\delta 2$	0.4998	0.0011	0.4912	0.0075	0.5020	0.0033	0.5009	0.0026
$O_{\alpha=-0.4}$	$\delta 3$	0.4764	0.0072	0.4694	0.0033	0.5233	0.0114	0.5207	0.0066
$O_{\alpha=-0.4}$	$\delta 4$	0.4350	0.0266	0.3810	0.0210	0.5907	0.0637	0.5686	0.0226
$O_{\alpha=-0.4}$	$\delta 1$	0.4752	0.0401	0.4341	0.0340	0.5487	0.0723	0.5571	0.0420
$O_{\alpha=-0.4}$	$\delta 5$	0.5108	0.0321	0.4475	0.0438	0.5389	0.0341	0.5435	0.0427
$O_{\alpha=-0.5}$	$\delta 0$	0.4551	0.0115	0.4391	0.0078	0.5475	0.0220	0.5384	0.0118
$O_{\alpha=-0.5}$	$\delta 2$	0.4997	0.0013	0.4916	0.0074	0.5017	0.0028	0.5010	0.0028
$O_{\alpha=-0.5}$	$\delta 3$	0.4785	0.0075	0.4684	0.0032	0.5238	0.0123	0.5193	0.0068
$O_{\alpha=-0.5}$	$\delta 4$	0.4247	0.0349	0.3940	0.0164	0.5980	0.0623	0.5618	0.0314
$O_{\alpha=-0.5}$	$\delta 1$	0.4519	0.0387	0.4245	0.0359	0.5459	0.0627	0.5604	0.0409
$O_{\alpha=-0.5}$	$\delta 5$	0.5002	0.0313	0.4483	0.0436	0.5387	0.0337	0.5420	0.0430
$O_{\alpha=-0.6}$	$\delta 0$	0.4577	0.0145	0.4404	0.0079	0.5516	0.0233	0.5364	0.0124
$O_{\alpha=-0.6}$	$\delta 2$	0.4998	0.0012	0.4913	0.0075	0.5023	0.0035	0.5009	0.0027
$O_{\alpha=-0.6}$	$\delta 3$	0.4773	0.0065	0.4683	0.0036	0.5252	0.0114	0.5194	0.0061
$O_{\alpha=-0.6}$	$\delta 4$	0.4203	0.0307	0.3764	0.0155	0.5966	0.0602	0.5740	0.0248
$O_{\alpha=-0.6}$	$\delta 1$	0.4549	0.0430	0.4279	0.0353	0.5560	0.0602	0.5631	0.0410
$O_{\alpha=-0.6}$	$\delta 5$	0.5088	0.0323	0.4474	0.0434	0.5384	0.0340	0.5422	0.0432
$O_{\alpha=-0.7}$	$\delta 0$	0.4575	0.0118	0.4412	0.0101	0.5547	0.0233	0.5371	0.0108
$O_{\alpha=-0.7}$	$\delta 2$	0.4997	0.0014	0.4909	0.0075	0.5022	0.0034	0.5015	0.0032
$O_{\alpha=-0.7}$	$\delta 3$	0.4780	0.0078	0.4676	0.0052	0.5249	0.0103	0.5188	0.0072
$O_{\alpha=-0.7}$	$\delta 4$	0.4164	0.0261	0.3884	0.0159	0.5967	0.0589	0.5743	0.0222
$O_{\alpha=-0.7}$	$\delta 1$	0.4688	0.0374	0.4291	0.0349	0.5638	0.0588	0.5628	0.0403
$O_{\alpha=-0.7}$	$\delta 5$	0.5035	0.0316	0.4479	0.0438	0.5378	0.0346	0.5423	0.0430
$O_{\alpha=-0.8}$	$\delta 0$	0.4574	0.0157	0.4349	0.0076	0.5454	0.0232	0.5390	0.0138
$O_{\alpha=-0.8}$	$\delta 2$	0.4998	0.0011	0.4899	0.0078	0.5024	0.0037	0.5009	0.0026

Table 25: Results for state 2 (S2)

F	δ	A_1		A_2		A_3		A_4	
		mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
$O_{\alpha=-0.8}$	$\delta 3$	0.4784	0.0079	0.4651	0.0047	0.5232	0.0120	0.5188	0.0072
$O_{\alpha=-0.8}$	$\delta 4$	0.4161	0.0272	0.3930	0.0184	0.6065	0.0602	0.5770	0.0218
$O_{\alpha=-0.8}$	$\delta 1$	0.4690	0.0367	0.4321	0.0343	0.5665	0.0541	0.5651	0.0401
$O_{\alpha=-0.8}$	$\delta 5$	0.5099	0.0318	0.4455	0.0444	0.5371	0.0339	0.5408	0.0442
$O_{\alpha=-0.9}$	$\delta 0$	0.4570	0.0142	0.4350	0.0071	0.5440	0.0250	0.5404	0.0131
$O_{\alpha=-0.9}$	$\delta 2$	0.4998	0.0012	0.4895	0.0079	0.5029	0.0042	0.5010	0.0028
$O_{\alpha=-0.9}$	$\delta 3$	0.4813	0.0079	0.4704	0.0027	0.5260	0.0113	0.5192	0.0066
$O_{\alpha=-0.9}$	$\delta 4$	0.4439	0.0327	0.3735	0.0126	0.6046	0.0641	0.5676	0.0279
$O_{\alpha=-0.9}$	$\delta 1$	0.4669	0.0389	0.4276	0.0353	0.5612	0.0515	0.5614	0.0410
$O_{\alpha=-0.9}$	$\delta 5$	0.5062	0.0312	0.4471	0.0441	0.5397	0.0338	0.5432	0.0428
$O_{\alpha=-1}$	$\delta 0$	0.4656	0.0137	0.4382	0.0121	0.5556	0.0246	0.5409	0.0111
$O_{\alpha=-1}$	$\delta 2$	0.4995	0.0018	0.4894	0.0078	0.5032	0.0045	0.5017	0.0038
$O_{\alpha=-1}$	$\delta 3$	0.4776	0.0065	0.4674	0.0028	0.5226	0.0110	0.5218	0.0060
$O_{\alpha=-1}$	$\delta 4$	0.4202	0.0427	0.3825	0.0120	0.5906	0.0652	0.5685	0.0357
$O_{\alpha=-1}$	$\delta 1$	0.4658	0.0384	0.4317	0.0353	0.5522	0.0531	0.5631	0.0403
$O_{\alpha=-1}$	$\delta 5$	0.5062	0.0318	0.4472	0.0440	0.5392	0.0345	0.5424	0.0432
$O_{\alpha=0.1}$	$\delta 0$	0.4597	0.0144	0.4373	0.0071	0.5431	0.0225	0.5362	0.0125
$O_{\alpha=0.1}$	$\delta 2$	0.4999	0.0008	0.4919	0.0073	0.5009	0.0016	0.5005	0.0019
$O_{\alpha=0.1}$	$\delta 3$	0.4784	0.0094	0.4657	0.0040	0.5226	0.0102	0.5178	0.0085
$O_{\alpha=0.1}$	$\delta 4$	0.4593	0.0272	0.3779	0.0190	0.5903	0.0652	0.5574	0.0241
$O_{\alpha=0.1}$	$\delta 1$	0.4610	0.0456	0.4274	0.0344	0.5385	0.0774	0.5628	0.0410
$O_{\alpha=0.1}$	$\delta 5$	0.5022	0.0306	0.4461	0.0435	0.5352	0.0337	0.5449	0.0422
$O_{\alpha=0.2}$	$\delta 0$	0.4590	0.0190	0.4308	0.0116	0.5435	0.0220	0.5343	0.0176
$O_{\alpha=0.2}$	$\delta 2$	0.4999	0.0009	0.4924	0.0069	0.5011	0.0019	0.5007	0.0021
$O_{\alpha=0.2}$	$\delta 3$	0.4783	0.0074	0.4693	0.0037	0.5238	0.0088	0.5189	0.0065
$O_{\alpha=0.2}$	$\delta 4$	0.4333	0.0331	0.3971	0.0181	0.5974	0.0600	0.5566	0.0293
$O_{\alpha=0.2}$	$\delta 1$	0.4755	0.0407	0.4267	0.0354	0.5459	0.0776	0.5615	0.0410
$O_{\alpha=0.2}$	$\delta 5$	0.5145	0.0330	0.4471	0.0438	0.5400	0.0355	0.5426	0.0430
$O_{\alpha=0.3}$	$\delta 0$	0.4607	0.0197	0.4337	0.0073	0.5395	0.0232	0.5342	0.0180
$O_{\alpha=0.3}$	$\delta 2$	0.4999	0.0007	0.4935	0.0067	0.5010	0.0017	0.5005	0.0018
$O_{\alpha=0.3}$	$\delta 3$	0.4779	0.0085	0.4687	0.0040	0.5218	0.0105	0.5183	0.0078
$O_{\alpha=0.3}$	$\delta 4$	0.4460	0.0319	0.3860	0.0236	0.5904	0.0635	0.5572	0.0271
$O_{\alpha=0.3}$	$\delta 1$	0.4602	0.0414	0.4328	0.0342	0.5419	0.0838	0.5568	0.0421
$O_{\alpha=0.3}$	$\delta 5$	0.5023	0.0316	0.4463	0.0434	0.5358	0.0336	0.5436	0.0427

Table 25: Results for state 2 (S2)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
$O_{\alpha=0.4}$	$\delta 0$	0.4574	0.0185	0.4377	0.0067	0.5416	0.0222	0.5359	0.0165
$O_{\alpha=0.4}$	$\delta 2$	0.4999	0.0008	0.4926	0.0069	0.5007	0.0013	0.5005	0.0018
$O_{\alpha=0.4}$	$\delta 3$	0.4768	0.0081	0.4670	0.0037	0.5202	0.0117	0.5199	0.0073
$O_{\alpha=0.4}$	$\delta 4$	0.4358	0.0343	0.3966	0.0167	0.5875	0.0579	0.5620	0.0303
$O_{\alpha=0.4}$	$\delta 1$	0.4644	0.0461	0.4312	0.0334	0.5420	0.0824	0.5578	0.0427
$O_{\alpha=0.4}$	$\delta 5$	0.5107	0.0323	0.4465	0.0435	0.5384	0.0351	0.5412	0.0434
$O_{\alpha=0.5}$	$\delta 0$	0.4564	0.0177	0.4366	0.0102	0.5389	0.0231	0.5361	0.0160
$O_{\alpha=0.5}$	$\delta 2$	0.4999	0.0007	0.4936	0.0068	0.5006	0.0012	0.5004	0.0017
$O_{\alpha=0.5}$	$\delta 3$	0.4789	0.0087	0.4684	0.0047	0.5233	0.0117	0.5188	0.0074
$O_{\alpha=0.5}$	$\delta 4$	0.4362	0.0329	0.3768	0.0172	0.5712	0.0612	0.5564	0.0301
$O_{\alpha=0.5}$	$\delta 1$	0.4608	0.0432	0.4330	0.0337	0.5407	0.0847	0.5564	0.0429
$O_{\alpha=0.5}$	$\delta 5$	0.5101	0.0324	0.4457	0.0435	0.5359	0.0352	0.5430	0.0427
$O_{\alpha=0.6}$	$\delta 0$	0.4580	0.0185	0.4379	0.0090	0.5397	0.0250	0.5351	0.0167
$O_{\alpha=0.6}$	$\delta 2$	0.4999	0.0006	0.4932	0.0069	0.5008	0.0014	0.5003	0.0015
$O_{\alpha=0.6}$	$\delta 3$	0.4777	0.0102	0.4679	0.0034	0.5206	0.0102	0.5186	0.0090
$O_{\alpha=0.6}$	$\delta 4$	0.4412	0.0301	0.3900	0.0164	0.5948	0.0567	0.5527	0.0279
$O_{\alpha=0.6}$	$\delta 1$	0.4619	0.0482	0.4274	0.0339	0.5331	0.0834	0.5612	0.0429
$O_{\alpha=0.6}$	$\delta 5$	0.5045	0.0316	0.4459	0.0434	0.5321	0.0342	0.5438	0.0424
$O_{\alpha=0.7}$	$\delta 0$	0.4612	0.0224	0.4373	0.0067	0.5401	0.0222	0.5334	0.0189
$O_{\alpha=0.7}$	$\delta 2$	0.4999	0.0006	0.4937	0.0066	0.5003	0.0006	0.5004	0.0016
$O_{\alpha=0.7}$	$\delta 3$	0.4757	0.0094	0.4677	0.0036	0.5184	0.0101	0.5208	0.0082
$O_{\alpha=0.7}$	$\delta 4$	0.4732	0.0377	0.3843	0.0128	0.5923	0.0617	0.5520	0.0309
$O_{\alpha=0.7}$	$\delta 1$	0.4642	0.0402	0.4314	0.0337	0.5389	0.0885	0.5566	0.0409
$O_{\alpha=0.7}$	$\delta 5$	0.5094	0.0320	0.4462	0.0434	0.5321	0.0349	0.5448	0.0423
$O_{\alpha=0.8}$	$\delta 0$	0.4617	0.0201	0.4408	0.0095	0.5437	0.0207	0.5323	0.0181
$O_{\alpha=0.8}$	$\delta 2$	0.5000	0.0006	0.4934	0.0067	0.5002	0.0004	0.5003	0.0014
$O_{\alpha=0.8}$	$\delta 3$	0.4786	0.0100	0.4694	0.0036	0.5219	0.0100	0.5179	0.0091
$O_{\alpha=0.8}$	$\delta 4$	0.4720	0.0388	0.3811	0.0122	0.5842	0.0609	0.5506	0.0335
$O_{\alpha=0.8}$	$\delta 1$	0.4748	0.0400	0.4310	0.0348	0.5460	0.0954	0.5562	0.0391
$O_{\alpha=0.8}$	$\delta 5$	0.5037	0.0326	0.4461	0.0433	0.5298	0.0344	0.5393	0.0447
$O_{\alpha=0.9}$	$\delta 0$	0.4571	0.0139	0.4431	0.0070	0.5480	0.0211	0.5374	0.0121
$O_{\alpha=0.9}$	$\delta 2$	0.5000	0.0005	0.4938	0.0066	0.5000	0.0001	0.5003	0.0013
$O_{\alpha=0.9}$	$\delta 3$	0.4792	0.0105	0.4678	0.0043	0.5220	0.0094	0.5174	0.0093
$O_{\alpha=0.9}$	$\delta 4$	0.4373	0.0444	0.3870	0.0094	0.5787	0.0560	0.5536	0.0393

Table 25: Results for state 2 (S2)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
$O_{\alpha=0.9}$	$\delta 1$	0.4688	0.0400	0.4298	0.0338	0.5342	0.0905	0.5560	0.0398
$O_{\alpha=0.9}$	$\delta 5$	0.4993	0.0304	0.4467	0.0432	0.5296	0.0349	0.5439	0.0421
$O_{\alpha=1}$	$\delta 0$	0.4598	0.0158	0.4434	0.0094	0.5467	0.0195	0.5355	0.0135
$O_{\alpha=1}$	$\delta 2$	0.5000	0.0005	0.4940	0.0066	0.5002	0.0003	0.5003	0.0012
$O_{\alpha=1}$	$\delta 3$	0.4806	0.0102	0.4679	0.0026	0.5207	0.0101	0.5164	0.0089
$O_{\alpha=1}$	$\delta 4$	0.4784	0.0338	0.3847	0.0114	0.5889	0.0591	0.5483	0.0286
$O_{\alpha=1}$	$\delta 1$	0.4654	0.0422	0.4291	0.0352	0.5360	0.0921	0.5549	0.0422
$O_{\alpha=1}$	$\delta 5$	0.5108	0.0326	0.4468	0.0431	0.5300	0.0347	0.5438	0.0425
O_{Div}	$\delta 0$	0.4721	0.0143	0.4369	0.0094	0.5231	0.0151	0.5223	0.0127
O_{Div}	$\delta 2$	0.5000	0.0004	0.4943	0.0064	0.5000	0.0002	0.5002	0.0010
O_{Div}	$\delta 3$	0.4799	0.0071	0.4639	0.0038	0.5130	0.0077	0.5165	0.0062
O_{Div}	$\delta 4$	0.4751	0.0234	0.3836	0.0220	0.5346	0.0430	0.5236	0.0167
O_{Div}	$\delta 1$	0.4864	0.0328	0.4131	0.0351	0.5548	0.0821	0.5565	0.0363
O_{Div}	$\delta 5$	0.4999	0.0294	0.4387	0.0422	0.5228	0.0377	0.5503	0.0396
GM	$\delta 0$	0.5000	0.0001	0.4947	0.0064	0.5000	0.0000	0.5000	0.0001
GM	$\delta 2$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
GM	$\delta 3$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
GM	$\delta 4$	0.4536	0.0205	0.4132	0.0139	0.5295	0.0260	0.5430	0.0208
GM	$\delta 1$	0.4748	0.0145	0.4262	0.0152	0.5474	0.0415	0.5368	0.0123
GM	$\delta 5$	0.4808	0.0111	0.4659	0.0061	0.5243	0.0146	0.5192	0.0097
HM	$\delta 0$	0.4689	0.0142	0.4337	0.0072	0.5252	0.0169	0.5229	0.0133
HM	$\delta 2$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
HM	$\delta 3$	0.4998	0.0007	0.4883	0.0049	0.4980	0.0117	0.5005	0.0014
HM	$\delta 4$	0.4574	0.0265	0.3831	0.0199	0.5359	0.0448	0.5329	0.0255
HM	$\delta 1$	0.4780	0.0329	0.4070	0.0353	0.5329	0.0856	0.5548	0.0339
HM	$\delta 5$	0.4971	0.0257	0.4436	0.0342	0.5164	0.0414	0.5442	0.0311
O_{mM}	$\delta 0$	0.4572	0.0137	0.4313	0.0102	0.5490	0.0188	0.5403	0.0124
O_{mM}	$\delta 2$	0.4997	0.0015	0.4885	0.0081	0.5039	0.0053	0.5014	0.0031
O_{mM}	$\delta 3$	0.4791	0.0071	0.4638	0.0023	0.5223	0.0119	0.5191	0.0073
O_{mM}	$\delta 4$	0.4486	0.0307	0.3858	0.0137	0.6018	0.0691	0.5695	0.0218
O_{mM}	$\delta 1$	0.4663	0.0406	0.4243	0.0363	0.5613	0.0475	0.5617	0.0413
O_{mM}	$\delta 5$	0.5092	0.0320	0.4460	0.0441	0.5400	0.0342	0.5424	0.0432
O_B	$\delta 0$	0.4752	0.0150	0.4358	0.0072	0.5434	0.0154	0.5226	0.0128
O_B	$\delta 2$	0.5000	0.0005	0.4933	0.0067	0.5001	0.0002	0.5003	0.0013

Table 25: Results for state 2 (S2)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
O_B	$\delta 3$	0.4801	0.0062	0.4660	0.0039	0.5188	0.0092	0.5172	0.0054
O_B	$\delta 4$	0.4641	0.0215	0.3858	0.0192	0.5694	0.0505	0.5473	0.0186
O_B	$\delta 1$	0.4779	0.0374	0.4202	0.0338	0.5321	0.0861	0.5628	0.0373
O_B	$\delta 5$	0.5020	0.0308	0.4444	0.0429	0.5275	0.0357	0.5463	0.0414
O_{RS}	$\delta 0$	0.4999	0.0008	0.4922	0.0071	0.5016	0.0026	0.5005	0.0019
O_{RS}	$\delta 2$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
O_{RS}	$\delta 3$	0.5000	0.0000	0.5000	0.0003	0.5000	0.0000	0.5000	0.0000
O_{RS}	$\delta 4$	0.4576	0.0225	0.4085	0.0122	0.5564	0.0222	0.5444	0.0229
O_{RS}	$\delta 1$	0.4566	0.0176	0.4413	0.0053	0.5473	0.0221	0.5360	0.0155
O_{RS}	$\delta 5$	0.4773	0.0075	0.4672	0.0041	0.5241	0.0098	0.5203	0.0067
Sin	$\delta 0$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
Sin	$\delta 2$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
Sin	$\delta 3$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
Sin	$\delta 4$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
Sin	$\delta 1$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
Sin	$\delta 5$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
T_P	$\delta 0$	0.4566	0.0176	0.4413	0.0053	0.5473	0.0221	0.5360	0.0155
T_P	$\delta 2$	0.4999	0.0008	0.4922	0.0071	0.5016	0.0026	0.5005	0.0019
T_P	$\delta 3$	0.4773	0.0075	0.4672	0.0041	0.5241	0.0098	0.5203	0.0067
T_P	$\delta 4$	0.4198	0.0396	0.3782	0.0190	0.5718	0.0605	0.5719	0.0360
T_P	$\delta 1$	0.4669	0.0375	0.4231	0.0367	0.5498	0.0772	0.5579	0.0398
T_P	$\delta 5$	0.5000	0.0310	0.4476	0.0436	0.5368	0.0336	0.5431	0.0427
T_{DP}	$\delta 0$	0.0000	0.0000	0.0000	0.0000	0.3524	0.0961	0.0000	0.0000
T_{DP}	$\delta 2$	0.0000	0.0000	0.0000	0.0000	0.3874	0.0830	0.0000	0.0000
T_{DP}	$\delta 3$	0.0000	0.0000	0.0000	0.0000	0.4158	0.0859	0.0000	0.0000
T_{DP}	$\delta 4$	0.0000	0.0000	0.0000	0.0000	0.2468	0.1376	0.0000	0.0000
T_{DP}	$\delta 1$	0.0000	0.0000	0.0000	0.0000	0.2654	0.1398	0.0000	0.0000
T_{DP}	$\delta 5$	0.0000	0.0000	0.0000	0.0000	0.3703	0.1018	0.0000	0.0000
T_{HP}	$\delta 0$	0.4772	0.0207	0.4327	0.0081	0.5230	0.0174	0.5217	0.0144
T_{HP}	$\delta 2$	0.5000	0.0003	0.4948	0.0063	0.5000	0.0000	0.5001	0.0005
T_{HP}	$\delta 3$	0.4885	0.0091	0.4656	0.0061	0.5263	0.0109	0.5160	0.0083
T_{HP}	$\delta 4$	0.4538	0.0255	0.3798	0.0208	0.5263	0.0433	0.5363	0.0241
T_{HP}	$\delta 1$	0.4844	0.0355	0.4067	0.0365	0.5330	0.0863	0.5554	0.0343
T_{HP}	$\delta 5$	0.5023	0.0289	0.4363	0.0414	0.5202	0.0420	0.5479	0.0384

Table 25: Results for state 2 (S2)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
T_L	$\delta 0$	0.4557	0.0058	0.4382	0.0099	0.5640	0.0138	0.5382	0.0106
T_L	$\delta 2$	0.4994	0.0021	0.4840	0.0089	0.5034	0.0046	0.5022	0.0041
T_L	$\delta 3$	0.4784	0.0024	0.4710	0.0063	0.5335	0.0079	0.5198	0.0042
T_L	$\delta 4$	0.4089	0.0123	0.3734	0.0231	0.6323	0.0310	0.5790	0.0204
T_L	$\delta 1$	0.5496	0.0585	0.4374	0.0433	0.6609	0.0354	0.5426	0.0473
T_L	$\delta 5$	0.5179	0.0336	0.4461	0.0454	0.5380	0.0379	0.5419	0.0440
T_M	$\delta 0$	0.4832	0.0147	0.4344	0.0093	0.5170	0.0175	0.5159	0.0076
T_M	$\delta 2$	0.5000	0.0002	0.4952	0.0060	0.5000	0.0000	0.5000	0.0002
T_M	$\delta 3$	0.4800	0.0092	0.4576	0.0047	0.4984	0.0107	0.5157	0.0081
T_M	$\delta 4$	0.4796	0.0148	0.3787	0.0206	0.5011	0.0402	0.5128	0.0130
T_M	$\delta 1$	0.4677	0.0327	0.3924	0.0385	0.5434	0.0798	0.5649	0.0352
T_M	$\delta 5$	0.4884	0.0277	0.4316	0.0411	0.5144	0.0434	0.5577	0.0367
T_{NM}	$\delta 0$	0.4591	0.0095	0.4496	0.0150	0.5727	0.0258	0.5362	0.0113
T_{NM}	$\delta 2$	0.4997	0.0016	0.4925	0.0070	0.5015	0.0024	0.5012	0.0031
T_{NM}	$\delta 3$	0.4795	0.0048	0.4744	0.0082	0.5366	0.0130	0.5178	0.0065
T_{NM}	$\delta 4$	0.5548	0.0655	0.3937	0.0256	0.6873	0.0453	0.5597	0.0146
T_{NM}	$\delta 1$	0.4684	0.0363	0.4434	0.0396	0.6209	0.0389	0.5469	0.0451
T_{NM}	$\delta 5$	0.5123	0.0321	0.4478	0.0454	0.5356	0.0374	0.5417	0.0439
$F_{BD1,q=0.1}$	$\delta 0$	0.4547	0.0410	0.3762	0.0433	0.5240	0.0937	0.5831	0.0345
$F_{BD1,q=0.1}$	$\delta 2$	0.5000	0.0000	0.4957	0.0058	0.5000	0.0000	0.5000	0.0000
$F_{BD1,q=0.1}$	$\delta 3$	0.4772	0.0206	0.4368	0.0223	0.5123	0.0476	0.5420	0.0175
$F_{BD1,q=0.1}$	$\delta 4$	0.4114	0.0800	0.2703	0.0778	0.5439	0.1757	0.6599	0.0654
$F_{BD1,q=0.1}$	$\delta 1$	0.4594	0.0832	0.3267	0.0845	0.5251	0.2127	0.6473	0.0705
$F_{BD1,q=0.1}$	$\delta 5$	0.4941	0.0376	0.4248	0.0479	0.5019	0.1076	0.5688	0.0384
$F_{BD1,q=0.2}$	$\delta 0$	0.4547	0.0410	0.3762	0.0433	0.5240	0.0937	0.5831	0.0345
$F_{BD1,q=0.2}$	$\delta 2$	0.5000	0.0000	0.4957	0.0058	0.5000	0.0000	0.5000	0.0000
$F_{BD1,q=0.2}$	$\delta 3$	0.4772	0.0206	0.4368	0.0223	0.5123	0.0476	0.5420	0.0175
$F_{BD1,q=0.2}$	$\delta 4$	0.4111	0.0793	0.2693	0.0752	0.5446	0.1720	0.6603	0.0646
$F_{BD1,q=0.2}$	$\delta 1$	0.4594	0.0832	0.3267	0.0845	0.5251	0.2127	0.6473	0.0705
$F_{BD1,q=0.2}$	$\delta 5$	0.4941	0.0376	0.4248	0.0479	0.5019	0.1076	0.5688	0.0384
$F_{BD1,q=0.3}$	$\delta 0$	0.4547	0.0410	0.3760	0.0430	0.5247	0.0927	0.5831	0.0345
$F_{BD1,q=0.3}$	$\delta 2$	0.5000	0.0000	0.4957	0.0058	0.5000	0.0000	0.5000	0.0000
$F_{BD1,q=0.3}$	$\delta 3$	0.4772	0.0206	0.4368	0.0223	0.5123	0.0476	0.5420	0.0175
$F_{BD1,q=0.3}$	$\delta 4$	0.4102	0.0772	0.2704	0.0731	0.5454	0.1667	0.6579	0.0637

Table 25: Results for state 2 (S2)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
$F_{BD1,q=0.3}$	$\delta 1$	0.4594	0.0832	0.3267	0.0845	0.5251	0.2127	0.6473	0.0705
$F_{BD1,q=0.3}$	$\delta 5$	0.4941	0.0376	0.4248	0.0479	0.5019	0.1076	0.5688	0.0384
$F_{BD1,q=0.4}$	$\delta 0$	0.4547	0.0410	0.3756	0.0419	0.5258	0.0909	0.5831	0.0345
$F_{BD1,q=0.4}$	$\delta 2$	0.5000	0.0000	0.4957	0.0058	0.5000	0.0000	0.5000	0.0000
$F_{BD1,q=0.4}$	$\delta 3$	0.4772	0.0206	0.4368	0.0223	0.5123	0.0476	0.5420	0.0175
$F_{BD1,q=0.4}$	$\delta 4$	0.4091	0.0751	0.2758	0.0731	0.5461	0.1612	0.6500	0.0645
$F_{BD1,q=0.4}$	$\delta 1$	0.4594	0.0832	0.3267	0.0845	0.5251	0.2127	0.6473	0.0705
$F_{BD1,q=0.4}$	$\delta 5$	0.4941	0.0376	0.4248	0.0479	0.5019	0.1076	0.5688	0.0384
$F_{BD1,q=0.5}$	$\delta 0$	0.4547	0.0410	0.3751	0.0408	0.5266	0.0888	0.5831	0.0345
$F_{BD1,q=0.5}$	$\delta 2$	0.5000	0.0000	0.4957	0.0058	0.5000	0.0000	0.5000	0.0000
$F_{BD1,q=0.5}$	$\delta 3$	0.4772	0.0206	0.4368	0.0223	0.5123	0.0476	0.5420	0.0175
$F_{BD1,q=0.5}$	$\delta 4$	0.4082	0.0731	0.2801	0.0729	0.5468	0.1564	0.6439	0.0648
$F_{BD1,q=0.5}$	$\delta 1$	0.4594	0.0832	0.3267	0.0845	0.5251	0.2127	0.6473	0.0705
$F_{BD1,q=0.5}$	$\delta 5$	0.4941	0.0376	0.4248	0.0479	0.5019	0.1076	0.5688	0.0384
$F_{BD1,q=0.6}$	$\delta 0$	0.4547	0.0409	0.3747	0.0397	0.5272	0.0869	0.5832	0.0344
$F_{BD1,q=0.6}$	$\delta 2$	0.5000	0.0000	0.4957	0.0058	0.5000	0.0000	0.5000	0.0000
$F_{BD1,q=0.6}$	$\delta 3$	0.4772	0.0206	0.4368	0.0223	0.5123	0.0476	0.5420	0.0175
$F_{BD1,q=0.6}$	$\delta 4$	0.4075	0.0715	0.2833	0.0725	0.5475	0.1525	0.6393	0.0649
$F_{BD1,q=0.6}$	$\delta 1$	0.4594	0.0832	0.3267	0.0845	0.5251	0.2127	0.6473	0.0705
$F_{BD1,q=0.6}$	$\delta 5$	0.4941	0.0376	0.4248	0.0479	0.5019	0.1076	0.5688	0.0384
$F_{BD1,q=0.7}$	$\delta 0$	0.4546	0.0407	0.3744	0.0389	0.5277	0.0853	0.5833	0.0342
$F_{BD1,q=0.7}$	$\delta 2$	0.5000	0.0000	0.4957	0.0058	0.5000	0.0000	0.5000	0.0000
$F_{BD1,q=0.7}$	$\delta 3$	0.4772	0.0206	0.4368	0.0223	0.5123	0.0476	0.5420	0.0175
$F_{BD1,q=0.7}$	$\delta 4$	0.4070	0.0702	0.2857	0.0722	0.5480	0.1495	0.6360	0.0649
$F_{BD1,q=0.7}$	$\delta 1$	0.4594	0.0832	0.3267	0.0845	0.5251	0.2126	0.6473	0.0705
$F_{BD1,q=0.7}$	$\delta 5$	0.4941	0.0376	0.4248	0.0479	0.5019	0.1076	0.5688	0.0384
$F_{BD1,q=0.8}$	$\delta 0$	0.4545	0.0405	0.3741	0.0383	0.5280	0.0841	0.5834	0.0340
$F_{BD1,q=0.8}$	$\delta 2$	0.5000	0.0000	0.4957	0.0058	0.5000	0.0000	0.5000	0.0000
$F_{BD1,q=0.8}$	$\delta 3$	0.4772	0.0206	0.4368	0.0223	0.5123	0.0476	0.5420	0.0175
$F_{BD1,q=0.8}$	$\delta 4$	0.4066	0.0692	0.2873	0.0720	0.5484	0.1473	0.6337	0.0649
$F_{BD1,q=0.8}$	$\delta 1$	0.4594	0.0832	0.3267	0.0845	0.5252	0.2126	0.6473	0.0705
$F_{BD1,q=0.8}$	$\delta 5$	0.4941	0.0376	0.4248	0.0479	0.5019	0.1076	0.5688	0.0384
$F_{BD1,q=0.9}$	$\delta 0$	0.4544	0.0403	0.3739	0.0379	0.5282	0.0833	0.5834	0.0338
$F_{BD1,q=0.9}$	$\delta 2$	0.5000	0.0000	0.4957	0.0058	0.5000	0.0000	0.5000	0.0000

Table 25: Results for state 2 (S2)

		A_1		A_2		A_3		A_4	
F	δ	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
$F_{BD1,q=0.9}$	$\delta 3$	0.4772	0.0206	0.4368	0.0223	0.5123	0.0476	0.5420	0.0175
$F_{BD1,q=0.9}$	$\delta 4$	0.4064	0.0685	0.2885	0.0718	0.5487	0.1457	0.6321	0.0648
$F_{BD1,q=0.9}$	$\delta 1$	0.4594	0.0832	0.3267	0.0845	0.5252	0.2125	0.6473	0.0705
$F_{BD1,q=0.9}$	$\delta 5$	0.4941	0.0376	0.4248	0.0479	0.5019	0.1076	0.5688	0.0384
$F_{BD1,q=1}$	$\delta 0$	0.4544	0.0402	0.3738	0.0376	0.5284	0.0827	0.5835	0.0337
$F_{BD1,q=1}$	$\delta 2$	0.5000	0.0000	0.4957	0.0058	0.5000	0.0000	0.5000	0.0000
$F_{BD1,q=1}$	$\delta 3$	0.4772	0.0206	0.4368	0.0223	0.5123	0.0476	0.5420	0.0175
$F_{BD1,q=1}$	$\delta 4$	0.4062	0.0680	0.2893	0.0717	0.5490	0.1446	0.6310	0.0648
$F_{BD1,q=1}$	$\delta 1$	0.4594	0.0832	0.3267	0.0845	0.5253	0.2125	0.6473	0.0705
$F_{BD1,q=1}$	$\delta 5$	0.4941	0.0376	0.4248	0.0479	0.5019	0.1076	0.5688	0.0384
F_{NA2}	$\delta 0$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
F_{NA2}	$\delta 2$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
F_{NA2}	$\delta 3$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
F_{NA2}	$\delta 4$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
F_{NA2}	$\delta 1$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
F_{NA2}	$\delta 5$	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000	0.5000	0.0000
F_{NA}	$\delta 0$	0.4706	0.0155	0.4319	0.0086	0.5116	0.0196	0.5236	0.0148
F_{NA}	$\delta 2$	0.5000	0.0002	0.4952	0.0060	0.5000	0.0000	0.5000	0.0002
F_{NA}	$\delta 3$	0.4845	0.0086	0.4645	0.0049	0.5179	0.0098	0.5123	0.0082
F_{NA}	$\delta 4$	0.4564	0.0295	0.3784	0.0207	0.5126	0.0528	0.5330	0.0290
F_{NA}	$\delta 1$	0.4741	0.0326	0.3929	0.0378	0.5277	0.0818	0.5604	0.0349
F_{NA}	$\delta 5$	0.4889	0.0271	0.4322	0.0409	0.5095	0.0436	0.5576	0.0364

B Theoretical Framework

B.1 Interval Numbers and Fuzzy Sets

We begin this section with the definition of interval number.

Definition B.1.1. An interval number an object of the form $\mathbf{a} = [a^L, a^U]$. Where a^L is the lower bound and a^U is the upper bound.

A distance between two interval numbers can be calculated with:

Definition B.1.2. Let $\mathbf{a} = [a^L, a^U]$ and $\mathbf{b} = [b^L, b^U]$ be 2 interval numbers, then the euclidean distance between them is:

$$d(\mathbf{a}, \mathbf{b}) = \sqrt{\frac{1}{2}((a^L - b^L)^2 + (a^U - b^U)^2)}.$$

Definition B.1.3. A fuzzy set a in a universe X is characterized by a membership function $\phi_a : X \rightarrow [0, 1]$, such that $0 \leq \phi_a(x) \leq 1$. In mathematical notation:

$$a = \{ \langle x, \phi_a(x) \rangle \mid x \in X \}.$$

A fuzzy set can characterize a fuzzy number, for example a trapezoidal fuzzy number has the form:

Definition B.1.4. A fuzzy set a is a trapezoidal fuzzy number, denoted $a = (a_1; a_2; a_3; a_4)$, if its membership function, $\phi_a : \mathbb{R} \rightarrow [0, 1]$, is given by:

$$\phi_a(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 < x < a_2 \\ 1, & a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 < x < a_4 \\ 0, & \text{otherwise} \end{cases}$$

Distance between two trapezoidal fuzzy numbers can be calculated via:

Definition B.1.5. Let $a = (a_1; a_2; a_3; a_4)$ and $b = (b_1; b_2; b_3; b_4)$ be two trapezoidal fuzzy numbers. Then, a distance measure between them is:

$$d(a, b) = \sqrt{\frac{1}{4} \sum_{i=1}^4 (a_i - b_i)^2}$$

The transformation from a fuzzy number to a real number (or crisp number) is called defuzzification and one of the methods for trapezoidal fuzzy numbers is:

Definition B.1.6. Let $a = (a_1; a_2; a_3; a_4)$ be a trapezoidal fuzzy number, then the defuzzified value of a is:

$$m(a) = \frac{a_1 + a_2 + a_3 + a_4}{4}$$

B.2 Aggregation Functions

Aggregation functions (or operators) are functions that, given an inputs, result in a combination of those inputs as a single output. This output resulted by the aggregation function, in a way (which depends on the context), tries to represent all individuals inputs as a single value.

Definition B.2.1 (Aggregation Function). An aggregation function is a function of $n > 1$ arguments that maps the unit cube onto the unit interval, $f : [0, 1]^n \rightarrow [0, 1]$, such that:

i) $f(\mathbf{0}) = 0$ and $f(\mathbf{1}) = 1$; (Boundary conditions)

ii) If $\mathbf{x} \leq \mathbf{y}$ then $f(\mathbf{x}) \leq f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in [0, 1]^n$. (Monotonicity)

Aggregation functions are present in an enormous amount of areas and its problems. A few examples of these functions includes the arithmetic mean(AM), the product (PR) and the harmonic mean (HM):

$$AM(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n x_i$$

$$PR(\mathbf{x}) = \prod_{i=1}^n x_i$$

$$HM(\mathbf{x}) = n \left(\sum_{i=1}^n \frac{1}{x_i} \right)^{-1}, \text{ where } x_i^{-1} = 0, \text{ if } x_i = 0$$

B.2.1 Properties

Here we present some useful properties of aggregation functions that will be convenient later.

Definition B.2.2 (Idempotent). An aggregation operator f is said to be idempotent if for all $\mathbf{x} = (t, \dots, t)$, where $t \in [0, 1]$, it holds $f(\mathbf{x}) = t$.

Due to the monotonicity property of f , the idempotency is equivalent to f be an averaging aggregation function (see section B.2.2).

For example, the arithmetic mean function is idempotent: if $\mathbf{x} = (t, \dots, t)$, where $|\mathbf{x}| = n$, we have $AM(\mathbf{x}) = n^{-1} \sum t = n^{-1} nt = t$. On the other hand, not all aggregation functions are idempotent. Take the product defined above, it is not idempotent as $PR(\mathbf{x}) = \prod t = t^n \neq t$, for $n > 1$.

Definition B.2.3 (Symmetry/Commutative). The function f is said to be symmetric if it does not depend on the permutation of its arguments:

$$f(\mathbf{x}) = f(x_{P(1)}, x_{P(2)}, \dots, x_{P(n)})$$

for all $\mathbf{x} \in [0, 1]$ and any permutation $P = (P(1), P(2), \dots, P(n))$ of $(1, 2, \dots, n)$.

Sometimes a permutation $P = (P(1), \dots, P(i), \dots, P(n))$ will be written as $((1), \dots, (i), \dots, (n))$ to simplify equations.

Commutativity is not guaranteed for aggregation functions. A common example of this is the weighted arithmetic mean (WAM): let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{w} = (w_1, \dots, w_n)$, then:

$$\text{WAM}(\mathbf{x}) = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n x_i} \quad (3)$$

Here, each w_i is associated with a x_i value. So, if we permute $\mathbf{x}$, WAM may not give the same output as the non-permuted vector.

To some of the problems shown later the order of the elements will be quite important, so we will introduce the notation for those cases:

Definition B.2.4 (Notation). Let $\mathbf{x}$ be a $\mathbb{R}^n$ vector and P a permutation, then we denote:

- i) $\mathbf{x} \nearrow$ if $\mathbf{x}$ is permuted in a non-decreasing order, i.e., $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$;
- ii) $\mathbf{x} \searrow$ if $\mathbf{x}$ is permuted in a non-increasing order, i.e., $x_{(1)} \geq x_{(2)} \geq \dots \geq x_{(n)}$.

Definition B.2.5 (Strict Monotonicity). We say that f is strictly monotonic if $\mathbf{x} \leq \mathbf{y}$, with $\mathbf{x} \neq \mathbf{y}$ then $f(\mathbf{x}) < f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in [0, 1]^n$. That is, at least one $y_i > x_i$, where $y_i \in \mathbf{y}$ and $x_i \in \mathbf{x}$.

Definition B.2.6 (Neutral Element). An element $e \in [0, 1]$ is called neutral element for the operator f if for all $t \in [0, 1]$ we have $f(e, \dots, e, t, e, \dots, e) = t$

Definition B.2.7 (Absorbing Element). An element $a \in [0, 1]$ is said to be an absorbing element for f if $f(x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_n) = a$.

Definition B.2.8 (Associative). The operator f is associative $f(f(x_1, x_2), x_3) = f(x_1, f(x_2, x_3))$, for any $x_1, x_2, x_3 \in [0, 1]$.

B.2.2 Main aggregation functions classes

The aggregation operators can be categorized by its behavior. The main classes are: averaging, conjunctive, disjunctive and mixed.

Now it is presented the definition for each category.

Definition B.2.9 (Averaging Aggregation). A function f has averaging behavior if for all $\mathbf{x}$ it is bounded by

$$\min(\mathbf{x}) \leq f(\mathbf{x}) \leq \max(\mathbf{x})$$

An important averaging aggregation function is given by the Ordered Weighting Averaging (OWA). This aggregation function resembles the WAM function defined earlier, but now it is taken into account the order of the elements in $\mathbf{x}$.

Definition B.2.10 (Weighting vector). Let $\mathbf{w} \in [0, 1]^n$ be a vector, then $\mathbf{w}$ is called a weighting vector if $\sum_{i=1}^n w_i = 1$.

Therefore the OWA function is:

Definition B.2.11 (OWA). Given a weighting vector $\mathbf{w}$ the OWA aggregation function of $\mathbf{x}$ is:

$$\text{OWA}_{\mathbf{w}}(\mathbf{x}) = \langle \mathbf{w}, \mathbf{x}_{\searrow} \rangle = \sum_{i=1}^n w_i x_{(i)} \quad (4)$$

Definition B.2.12 (Conjunctive Aggregation). A function f has conjunctive behavior if for all $\mathbf{x}$ it is bounded by:

$$f(\mathbf{x}) \leq \min(\mathbf{x})$$

The most obvious conjunctive function is the one defined by $f(\mathbf{x}) = \min(\mathbf{x})$. Another example is the product aggregation function PR , when $\mathbf{x} \in [0, 1]^n$, we have that $PR(\mathbf{x}) = \prod x_i \leq \min(\mathbf{x})$.

Definition B.2.13 (Disjunctive Aggregation). A function f has disjunctive behavior if for all $\mathbf{x}$ it is bounded by:

$$f(\mathbf{x}) \geq \max(\mathbf{x})$$

Here, $f(\mathbf{x}) = \max(\mathbf{x})$ is the trivial example.

Definition B.2.14 (Mixed Aggregation). A function f has mixed behavior if it does not belong to any of the other classes.

B.3 Triangular Norms

Definition B.3.1 (Triangular Norm). A triangular norm (or t-norm) is a bi-variate aggregation function $T : [0, 1]^2 \rightarrow [0, 1]$, such that, for all $x, y, z \in [0, 1]$, T satisfies the following properties:

- T1) $T(x, y) = T(y, x)$; (Commutativity)
- T2) $T(x, T(y, z)) = T(T(x, y), z)$; (Associativity)
- T3) $T(x, y) \leq T(x, z)$ if $y \leq z$; (Monotonicity)
- T4) $T(x, 1) = x$. (Boundary condition)

Some examples of t-norms are:

- $T_M(x, y) = \min(x, y)$ (minimum)
- $T_P(x, y) = x \cdot y$ (product)
- $T_L(x, y) = \max(x + y - 1, 0)$ (Łukasiewicz t-norm)

Definition B.3.2 (Triangular t-conorm). A triangular conorm (or t-conorm) is a bi-variate aggregation function $S : [0, 1]^2 \rightarrow [0, 1]$, such that for all $x, y, z \in [0, 1]$, it satisfies:

- S1) $S(x, y) = S(y, x)$; (Commutativity)
- S2) $S(x, S(y, z)) = S(S(x, y), z)$; (Associativity)
- S3) $S(x, y) \leq S(x, z)$ if $y \leq z$; (Monotonicity)
- S4) $S(x, 0) = x$. (Boundary condition)

Some examples of t-conorms are:

- $S_M(x, y) = \max(x, y)$ (maximum)
- $S_P(x, y) = x + y - x \cdot y$ (probabilistic sum)
- $S_L(x, y) = \min(x + y, 1)$ (Łukasiewicz t-conorm, bounded sum)

B.4 Copula and Overlap Functions

Copulas are important functions in probability theory, as they function as a link between n -dimensional probability distributions and their one-dimensional marginal distribution. If we consider the domain of C to be $[0, 1]^2$, we have a 2-copula and, the definition is:

Definition B.4.1 (2-copula). A bivariate function $C : [0, 1]^2 \rightarrow [0, 1]$ is a copula if it satisfies the following conditions, for all $x_1, x_2, y_1, y_2 \in [0, 1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$:

$$\text{C1) } C(x_1, y_1) + C(x_2, y_2) \geq C(x_1, y_2) + C(x_2, y_1);$$

$$\text{C2) } C(x_1, 0) = C(0, x_1) = 0;$$

$$\text{C3) } C(x_1, 1) = C(1, x_1) = x_1$$

Some examples of copulas are [14]:

$$\begin{aligned} C_F(x, y) &= xy + x^2y(1-x)(1-y) \\ C_L(x, y) &= \max\{\min\{x, 2y^{-1}\}, x + y - 1\} \end{aligned}$$

Overlap functions were introduced by Bustince et al. [8] as a manner to deal with overlapping classes in classifications problems. Firstly used for object classification in image processing, now they are another useful tool where associativity is not required.

Definition B.4.2 (Overlap Function). A bivariate function $O : [0, 1]^2 \rightarrow [0, 1]$ is said to be an overlap function if it satisfies the following conditions:

$$\text{O1) } O(x, y) = O(y, x) \text{ for all } x, y \in [0, 1];$$

$$\text{O2) } O(x, y) = 0 \text{ if and only if } x = 0 \text{ or } y = 0;$$

$$\text{O3) } O(x, y) = 1 \text{ if and only if } x = y = 1;$$

$$\text{O4) } O \text{ is non-decreasing;}$$

$$\text{O5) } O \text{ is continuous;}$$

Examples of overlap functions are [14]:

$$\begin{aligned} O_{B,\alpha}(x, y) &= (T_M(x, y))^\alpha (T_P(x, y))^{1-\alpha}, \text{ where } \alpha \in [0, 1]; \\ O_\alpha &= xy(1 + \alpha(1-x)(1-y)), \alpha \in [-1, 1] \end{aligned}$$

B.5 Fuzzy measures and the Choquet Integral

Fuzzy measures are a generalization of measure theory where the additive property, $\mu(A \cup B) = \mu(A) + \mu(B)$, is weakened to a monotonicity property.

Definition B.5.1 (Fuzzy measure [34]). Let N be a finite set and 2^N be its power set. Then a function $\mu : 2^N \rightarrow [0, 1]$ is a fuzzy measure if, for all $X, Y \subseteq N$, the following conditions holds:

- 1) $\mu(\emptyset) = 0$ and $\mu(X) = 1$;
- 2) if $X \subseteq Y$, then $\mu(X) \leq \mu(Y)$.

If we have $\mu(X) = \mu(Y)$ whenever $|X| = |Y|$, we call μ a symmetric fuzzy measure. An common example of fuzzy measure is the power measure given by:

$$\begin{aligned} \mu : 2^N &\rightarrow [0, 1] \\ X &\rightarrow \left(\frac{|X|}{n} \right)^q, \text{ where } q > 0 \text{ and } n = |2^N|. \end{aligned} \quad (5)$$

Definition B.5.2 (Choquet Integral [10]). Let μ be a fuzzy measure. Then the Choquet integral of $\mathbf{x} \in [0, 1]^n$ with respect to μ is $\mathfrak{C} : [0, 1]^n \rightarrow [0, 1]$:

$$\begin{aligned} \mathfrak{C}_\mu(\mathbf{x}) &= \sum_{i=1}^n [x_{(i)} - x_{(i-1)}] \mu(A_{(i)}) \\ &= \sum_{i=1}^n x_{(i)} [\mu(A_{(i)}) - \mu(A_{(i+1)})] \end{aligned}$$

where P is a permutation on 2^N such that $\mathbf{x}_{\nearrow}$, with $x_{(0)} = 0$, $x_{(n+1)} = \emptyset$ and $A_{(i)} := \{(1), \dots, (n)\}$.

We can use the distributive law of multiplication in the Choquet integral to have the expanded Choquet integral:

$$\mathfrak{C}_\mu(\mathbf{x}) = \sum_{i=1}^n [x_{(i)}\mu(A_{(i)}) - x_{(i-1)}\mu(A_{(i)})] \quad (6)$$

In the works of Lucca et al. [27, 28, 29, 30], Dimuro et al. [15], they studied how one can replace the product operation in the Choquet integral for others operators, like copula and fusion functions. Additionally, a new type of operators were used: pre-aggregation functions (see next section), where they made quite an improvement on classification algorithms.

B.5.1 Generalizations of the Choquet Integral

When considering the monotonicity for a function f , a numerous quantity of functions are left behind. To overcome this problem, Bustince et al. [5] proposed an alternative to monotonicity called directional monotonicity, which describes monotonicity in a fixed direction, as follows:

Definition B.5.3. Let $\mathbf{r}$ be a real n -dimensional vector, such that $\mathbf{r} \neq \mathbf{0}$. A function $f : [0, 1]^n \rightarrow [0, 1]$ is said to be $\mathbf{r}$ -increasing if, for all $\mathbf{x} \in [0, 1]^n$ and for all $c > 0$ such that $\mathbf{x} + c\mathbf{r} \in [0, 1]^n$, it holds

$$f(\mathbf{x} + c\mathbf{r}) \geq f(\mathbf{x}) \quad (7)$$

From this definition, Lucca et al. [27] introduced pre-aggregation functions:

Definition B.5.4. A function $PA : [0, 1]^n \rightarrow [0, 1]$ is said to be a n -ary pre-aggregation function if the following condition holds:

(PA1) There exists $\mathbf{r} \in [0, 1]^n$, $\mathbf{r} \neq \mathbf{0}$, such that PA is $\mathbf{r}$ -increasing;

(PA2) $PA(\mathbf{0}) = 0$ and $PA(\mathbf{1}) = 1$.

There are quite a few functions which won't fit on the definition of aggregation functions but are useful and fits the pre-aggregation definition. For example:

Example B.5.1. The $Mode(\mathbf{x})$ function, which is used in statistics, is not monotonic increasing, as the minimum value can have more elements in a set. But if we look at the direction $\mathbf{1}$, it is an increasing function. Therefore, it is $\mathbf{1}$ -increasing and therefore is a pre-aggregation function.

From the definition of pre-aggregation functions, we can define new classes of the Choquet integral. A simple substitution of the product operator by a t-norm T creates a new class of Choquet integrals, namely C_T -integrals [27].

Definition B.5.5 (C_T -Integral). Let μ be a fuzzy measure and $T : [0, 1]^2 \rightarrow [0, 1]$ a t-norm. Then the $\mathfrak{C}_\mu^T : [0, 1]^2 \rightarrow [0, 1]$ integral of $\mathbf{x} \in [0, 1]^n$ is defined as

$$\mathfrak{C}_\mu^T(\mathbf{x}) = \sum_{i=1}^n T(x_{(i)} - x_{(i-1)}, \mu(A_{(i)})) \quad (8)$$

Instead of a t-norm T we can use a more general function F , with a small restriction, as follows:

Definition B.5.6 (C_F -Integral). Let μ be a fuzzy measure and $F : [0, 1]^2 \rightarrow [0, 1]$ a fusion function (any function such that $F : [0, 1]^n \rightarrow [0, 1]$) that is left 0-absorbent, that is, for

all $y \in [0, 1]$ we have $F(0, y) = 0$. Then $\mathfrak{C}_\mu^F : [0, 1]^2 \rightarrow [0, 1]$ integral is defined for all $\mathbf{x} \in [0, 1]^n$ by

$$\mathfrak{C}_\mu^F(\mathbf{x}) = \min \left\{ 1, \sum_{i=1}^n F(x_{(i)} - x_{(i-1)}, \mu(A_{(i)})) \right\} \quad (9)$$

Introduced by `luccaCFintegralsNewFamily2018`, the authors have shown that the $\mathfrak{C}_\mu^F$ is an 1-pre-aggregation function, whenever F is 0-absorbent and satisfies the right neutral element: for all $x \in [0, 1]$ it holds $F(x, 1) = x$.

On the above generalizations of the Choquet integral only one operation was modified. However, if we use its expanded form (Eq. (6)), we can obtain new classes of the Choquet integral. One of them is constructed by substituting the two products by a copula function [28]:

Definition B.5.7 (CC-Integral). Let μ be a fuzzy measure and C a 2-copula. Then the $\mathfrak{C}_\mu^C : [0, 1]^2 \rightarrow [0, 1]$ integral is defined for all $\mathbf{x} \in [0, 1]^n$ by

$$\mathfrak{C}_\mu^C(\mathbf{x}) = \sum_{i=1}^n C(x_{(i)}, \mu(A_{(i)})) - C(x_{(i-1)}, \mu(A_{(i)})) \quad (10)$$

B.5.2 d-Choquet integrals

Additionally to the previous generalizations of the Choquet integrals where the product of the points distance and the measure were substituted by a function, there are the d-Choquet integrals [6]. Such integrals use a restricted dissimilarity function δ instead of the points difference [4].

Definition B.5.8 ([4]). A function $\delta : [0, 1]^2 \rightarrow [0, 1]$ is called a restricted dissimilarity function if it satisfies the following conditions:

- (D1) $\delta(x, y) = \delta(y, x)$ for all $x, y \in [0, 1]$;
- (D2) $\delta(x, y) = 1$ if and only if $x = 0$ and $y = 0$ or $x = 1$ and $y = 0$;
- (D3) $\delta(x, y) = 0$ if and only if $x = y$;
- (D4) If $x \leq y \leq z$ then $\delta(x, y) \leq \delta(x, z)$ and $\delta(y, z) \leq \delta(x, z)$, for all $x, y, z \in [0, 1]$.

Definition B.5.9 (d-Choquet [6]). Let n be a positive integer and $\mu : 2^N \rightarrow [0, 1]$ be a fuzzy measure. And let $\delta : [0, 1]^2 \rightarrow [0, 1]$ be a restricted dissimilarity function. A n -ary discrete d-Choquet integral on $[0, 1]$ with respect to μ and δ is defined as a mapping $\mathfrak{C}_{\mu, \delta} : [0, 1]^n \rightarrow [0, n]$ such that

$$\mathfrak{C}_{\mu, \delta}(\mathbf{x}) = \sum_{i=1}^n \delta(x_{(i)}, x_{(i-1)}) \mu(A_{(i)}) \quad (11)$$

where $x_{(i)} \leq x_{(i+1)} \forall i = 0, \dots, n-1$, with $x_{(0)} = 0$ and $A_{(i)} = \{(i), \dots, (n)\}$.

B.6 TOPSIS

When dealing with decisions we are faced with a variety of options where we can choose one. That is why choosing is not always an easy task as we always search for the optimal solution, but not always absolute sure if we made the right choice after selecting one.

In this context, theory and methods were created to aid decision makers in making the better choice available. Decision Theory, an interdisciplinary topic, began to formulate, mathematically, the decision problem and research methods to solve them [32, 21].

Informally, when making a choice, we are analyzing various alternative with respect to some criteria. Each criterion will have a rating(or utility) for all the defined alternatives. Mathematically, we can define a set of alternatives $A = \{A_1, \dots, A_m\}$, where each A_i represents one alternative. A set of criteria $C = \{C_1, \dots, C_n\}$, where every criterion C_j is pre-defined. Therefore, the one making the decision (decision maker) can assign a value for each criterion C_j and alternative A_i with x_{ij} . This rating is what will be used for finding the best alternative from the given ones.

Additionally, the decision maker can also influence the decision by giving weights for each criterion. This is possible through a weighting vector $= (w_1, \dots, w_m)$, where each w_j corresponds to a criterion C_j .

From the definition above we can assemble the problem in a matrix, called decision matrix, where the columns represent the criteria and the lines represent the alternatives.

Definition B.6.1 (Decision Matrix). Given a set of m alternatives, denoted by $A_1, \dots, A_m$ and a set of n criteria, $C_1, \dots, C_n$, the matrix $D = [x_{ij}]_{(m \times n)}$ is called the decision matrix. Each x_{ij} indicates the rating of the alternative A_i with respect to the criteria C_j . Additionally, the set $= (w_1, \dots, w_n)$ is the weighting vector for each of the j criteria.

$$D = \begin{matrix} & C_1 & C_2 & \cdots & C_n \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} & \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \end{matrix} \quad (12)$$

The objective of the decision making methods is, given the values x_{ij} and w_j , to rank the alternatives based in the established criteria.

B.6.1 TOPSIS Method

A classical method for solving MADM problems is the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), that selects the best alternative based on the concept that it should have the shortest distance from the positive ideal solution (PIS) A^+ and the farthest distance from the negative ideal solution (NIS) A^- .

B.6.2 The TOPSIS Algorithm

The classical TOPSIS method is based on finding the alternative that has the minimum euclidean distance to the PIS and the maximum distance to the NIS. To this end, the algorithm firstly normalize and compose the decision matrix D with the weighting vector; secondly, the PIS and NIS solutions are selected; thirdly, the separation measure between each alternative and PIS, NIS are calculated; finally, the ranking is made with the non-crescent ordering of the relative closeness coefficients.

The algorithm is decribed in the following steps:

1) Let D be a decision matrix. First, the method construct the normalized decision matrix $R = [r_{ij}]_{m \times n}$ as follows:

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{k=1}^m x_{kj}^2}} \quad (13)$$

After having the normalized decision matrix R , we use of the weighting vector to give weights to the criteria C_j by calculating a new matrix $V = [v_{ij}]_{m \times n}$, where $v_{ij} = w_j \times r_{ij}$:

$$V = \begin{pmatrix} w_1 r_{11} & w_2 r_{12} & \cdots & w_n r_{1n} \\ w_1 r_{21} & w_2 r_{22} & \cdots & w_n r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_1 r_{m1} & w_2 r_{m2} & \cdots & w_n r_{mn} \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{pmatrix} \quad (14)$$

Therefore we can determine A^+ and A^- by:

$$A^+ = (v_1^+, \dots, v_n^+)$$

$$A^- = (v_1^-, \dots, v_n^-)$$

where

$$v_j^+ = \begin{cases} \max_i(v_{ij}), & \text{if it is a benefit criterion} \\ \min_i(v_{ij}), & \text{if it is a cost/loss criterion} \end{cases}, \quad (15)$$

$$v_j^- = \begin{cases} \min_i(v_{ij}), & \text{if it is a benefit criterion} \\ \max_i(v_{ij}), & \text{if it is a cost/loss criterion} \end{cases} \quad (16)$$

2) Calculate the separation measure for each alternative A_i , that is, calculate the euclidean distance of each alternative A_i to the PIS and NIS:

$$S_i^+ = \sqrt{\sum_{j=1}^n (v_{ij} - v_j^+)^2} \quad (17)$$

$$S_i^- = \sqrt{\sum_{j=1}^n (v_{ij} - v_j^-)^2} \quad (18)$$

3) Calculate the relative closeness coefficients to the ideal solution with the next equation. Since the separation measure are numbers, this will show the relation between distances to the PIS and NIS, consequently the greater the number the bigger the distance of the alternative to the NIS:

$$C_i^+ = \frac{S_i^-}{S_i^+ + S_i^-} \quad (19)$$

where $0 \leq C_i^+ \leq 1$ and $i = 1, 2, \dots, m$.

4) Finally, the optimal solution can be ranked based on the preference rank order of C_i^+ . That is, the best alternative c has the shortest distance to the ideal solution A^+ , i.e.,

$$c = \{A_\alpha \mid \alpha = \arg \max(C_i^+)\}. \quad (20)$$

B.6.3 Fuzzy Group TOPSIS

One drawback of the classical TOPSIS is the inability to deal with criteria that are interdependent. For example the price of a product may be dependent on the quality of such product. Additionally, there are decisions that must be made by groups of people, and not only one person. This problem has been worked on by a few studies for the classical and fuzzy TOPSIS [24, 39, 9].

Furthermore, the real world needs an appropriated way of dealing with its uncertainty. In this concept, Fuzzy Set Theory, arrive as an optimal tool to solve the problem.

In the works of Lourenzutti et al. [26], it is presented a method named Group Modular Choquet Random TOPSIS (GMC-RTOPSIS), where the issues presented in the above paragraphs are solved using fuzzy measures and the Choquet integral.

The group decision making problem is broken into modules, that is, separating each criterion into its module. Then each module is worked on separated, to be aggregated on the last step of the process.

On the classical TOPSIS we had a single matrix D representing the ratings, or utility, of each criterion for every alternative. For a group decision problem we define a D matrix for each of the decision maker. So, let $d = 1, \dots, k$ represent the decision maker, $A = \{A_1, \dots, A_m\}$ the m alternatives. $C^d = \{C_1^d, \dots, C_{n_d}^d\}$ the criteria set for each decision

maker; $\mathbf{C} = \{\mathbf{C}^1, \dots, \mathbf{C}^k\} = \{C_1, \dots, C_n\}$, where $n = n_1 + \dots + n_k$, the criteria set of all decision makers.

Additionally, we may have underlying factors that affect the performance of alternatives. Those effects can be random ($\mathbf{Y}_{rand}$), such as stochastic factors, or deterministic ($\mathbf{Y}_{det}$), such as time, and will be denoted by $\mathbf{Y} = (\mathbf{Y}_{rand}, \mathbf{Y}_{det})$. Also, note that $\mathbf{Y}_{rand}$ may dependent on $\mathbf{Y}_{det}$. Moreover, an fixed configuration of $\mathbf{Y}_{det}$ is called a state and, the set $\mathcal{X}$ is the set of all states.

From these definitions, we can represent a decision matrix for each decision maker d by:

$$DM^d = \begin{matrix} & C_1 & C_2 & \dots & C_n \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} & \begin{pmatrix} s_{11}^d(\mathbf{Y}^d) & s_{12}^d(\mathbf{Y}^d) & \dots & s_{1n}^d(\mathbf{Y}^d) \\ s_{21}^d(\mathbf{Y}^d) & s_{22}^d(\mathbf{Y}^d) & \dots & s_{2n}^d(\mathbf{Y}^d) \\ \vdots & \vdots & \ddots & \vdots \\ s_{m1}^d(\mathbf{Y}^d) & s_{m2}^d(\mathbf{Y}^d) & \dots & s_{mn}^d(\mathbf{Y}^d) \end{pmatrix} \end{matrix} \quad (21)$$

B.6.4 GMC-RTOPSIS Algorithm

In this section it is presented the procedure to calculate the best solution for the decision problem. The steps follows the classical TOPSIS in the sense of how the algorithm works. Although, since working with multiple types of inputs, it is better to work with each criterion separately. This will be called a module. Additionally, the separation measure is calculated using the Choquet integral. An overview of how the method works is presented in Fig. 3.

1. Each decision maker provide its decision matrix $DM^d(\mathbf{Y})$;
2. Normalize the matrices;
3. Select the PIS, $s_j^+(\mathbf{Y})$, and the NIS, $s_j^-(\mathbf{Y})$ based on Def. 15.
4. Calculate, for each module $C_j \in \mathbf{C}$, the distance measure $d_{ij}^+ = d(s_j^+(\mathbf{Y}), s_{ij}(\mathbf{Y}))$ and $d_{ij}^- = d(s_j^-(\mathbf{Y}), s_{ij}(\mathbf{Y}))$, observing that the measure is dependent on the type of data being processed.
5. Calculate the separation measure for each alternative with:

$$S_i^+(\mathbf{Y}) = \sqrt{\sum_{j=1}^n \left(d_{ij}^+\right)^2 \left(\varphi_{\mathbf{Y}}(\mathbf{C}_{(j)}^+) - \varphi_{\mathbf{Y}}(\mathbf{C}_{(j+1)}^+)\right)} \quad (22)$$

$$S_i^-(\mathbf{Y}) = \sqrt{\sum_{j=1}^n \left(d_{ij}^-\right)^2 \left(\varphi_{\mathbf{Y}}(\mathbf{C}_{(j)}^-) - \varphi_{\mathbf{Y}}(\mathbf{C}_{(j+1)}^-)\right)} \quad (23)$$

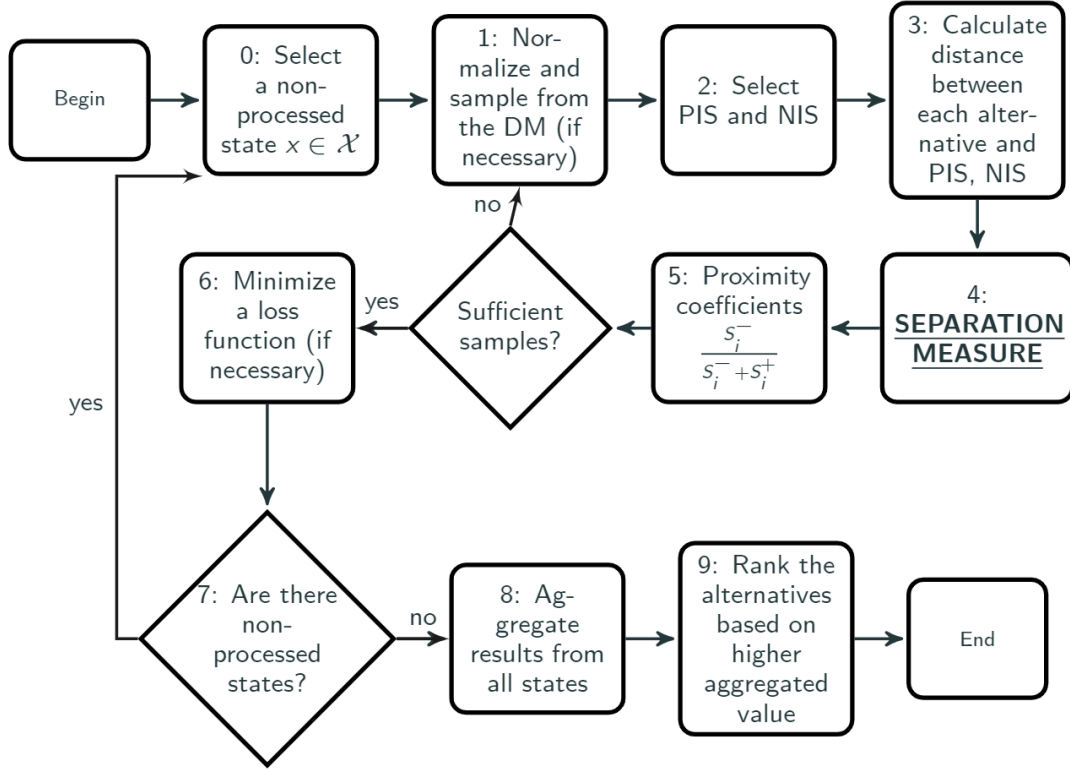


Figure 3: Diagram of the GMC-RTOPSIS algorithm.

where $d_{i \nearrow}$, $C_{(j)}$ is the corresponding criteria to $d_{i(j)}$, $C_{(j)} = \{C_{(j)}, C_{(j+1)}, \dots, C_{(n)}\}$ and $C_{n+1} = \emptyset$. Additionally, each state Y_{det} may have a different fuzzy measure;

6. Calculate the relative closeness coefficients to the ideal solution for each alternative:

$$CC_i(\mathbf{Y}) = \frac{S_i^-(\mathbf{Y})}{S_i^-(\mathbf{Y}) + S_i^+(\mathbf{Y})} \quad (24)$$

7. Since CC_i is a bootstrapped probability distribution, we need to find the closer distribution to CC_i . Thus, the rank of the alternatives use a risk function that minimizes the error in choosing one distribution over another:

$$\begin{aligned} \widehat{cc}_i &= \arg \min_c R(c) \\ &= \arg \min_c \int_{\mathbb{R}} L(c, CC_i(\mathbf{Y})) dF(CC_i(\mathbf{Y})) \end{aligned} \quad (25)$$

8. Obtain the final estimation $\widehat{cc}_i = f_{x \in \mathcal{X}}(\widehat{cc}_i(x))$, where f is an aggregation function that assembles the $\widehat{cc}_i$ from all states.

Observe that the steps 1-7 are necessary for each state $x \in \mathcal{X}$ independently. After, an aggregation function f can be used to obtain in final result. For example, $f = HM$, $f = AM$.